

Existence of weak solutions for a general porous medium equation with nonlocal pressure

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Abstract

We study the general nonlinear diffusion equation $u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u)$ that describes a flow through a porous medium which is driven by a nonlocal pressure. We consider constant parameters $m > 1$ and $0 < s < 1$, we assume that the solutions are non-negative and the problem is posed in the whole space. In this paper we prove existence of weak solutions for all integrable initial data $u_0 \geq 0$ and for all exponents $m > 1$ by developing a new approximating method that allows to treat the range $m \geq 3$ that could not be covered by previous works. We also consider as initial data any non-negative measure μ with finite mass. In passing from bounded initial data to measure data we make strong use of an L^1 - L^∞ smoothing effect and other functional inequalities. Finite speed of propagation is established for all $m \geq 2$, which implies the existence of free boundaries. The authors had already proved that finite propagation does not hold for $m < 2$.

Keywords: Nonlinear fractional diffusion, fractional Laplacian, existence of weak solutions, energy estimates, speed of propagation, smoothing effect, numerical simulations.

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1 Introduction

In this paper we study the following evolution equation of diffusive type with nonlocal effects

$$(1.1) \quad \begin{cases} \partial_t u = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s} u) & \text{for } x \in \mathbb{R}^N, t > 0, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

for $u = u(x, t)$, exponents $m > 1$, $0 < s < 1$, and space dimension $N \geq 1$. We will only consider nonnegative data and solutions $u_0, u \geq 0$ on physical grounds. The problem will be posed in the whole space, with $x \in \mathbb{R}^N$ and $t > 0$. Here $(-\Delta)^{-s}$ denotes the inverse of the fractional Laplacian operator as defined in [33].

Our aim is to construct weak solutions for all initial data $u_0 \in L^1(\mathbb{R}^N)$ and for all the stated range of parameters. Model (1.1) formally resembles the Porous Medium Equation $\partial_t u = \nabla \cdot (u^{m-1} \nabla u)$ when $s = 0$, but here we allow for a new dependence via the inverse fractional Laplacian operator, $\partial_t u = \nabla \cdot (u^{m-1} \nabla p)$ with $p = (-\Delta)^{-s} u$, which accounts for nonlocal effects in the diffusive process. We will call this intermediate variable p the pressure, though it is not in agreement with the usual PME convention unless $m = 2$.

The problem for $m = 2$ was studied by Caffarelli and Vázquez starting with [9, 10], followed by [7, 8, 11]. Our model is a particular case of the general equations proposed in [17, 18] in statistical physics, which take the form $u_t = \nabla \cdot (\sigma(u) \nabla \mathcal{L}(u))$. There is also a physical motivation in the theory of dislocations proposed by Head, that has been investigated by Biler, Karch and Monneau [3] for $m = 2$ in one space dimension. However, the extension of the dislocation model to several dimensions leads to a more complicated system that falls outside of the present investigation. Finally, we point out that the gradient flow structure for (1.1) with $m = 2$ has been recently developed in [22] using Wasserstein metrics in the style of [1]. Uniqueness is still an open problem for all these models in several space dimensions, but it holds for $N = 1$ according to [3]. There are recent uniqueness results if the initial data are very smooth, see [37]. They obtain unique local-in-time strong solutions in Besov spaces; thus, for initial data in $B_{1,\infty}^\alpha$ if $1/2 \leq s < 1$ and $\alpha > N + 1$ with $N \geq 2$.

Existence of constructed weak solutions for $m \in (1, 3)$ was proved by the same authors in [30, 32] under some extra decay conditions on the initial data. In that paper we employed a rather standard regularization of the singular operator by considering $K_\epsilon \star u \rightarrow |x|^{-(N-2s)} \star u = (-\Delta)^{-s} u$ where K_ϵ is a suitable smooth kernel. Energy estimates allowed us to obtain compactness, but only in the range of m . New methods seemed to be needed for the more degenerate case $m \geq 3$. A further discussion on this issue can be found in Section 6. The main step we take here in order to prove existence of weak solutions of (1.1) is a novel approximation method. This consists in interpreting model (1.1) in the form

$$u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-1} \mathcal{L} u).$$

Then we approximate the operator $\mathcal{L} = (-\Delta)^{1-s}$ by

$$\mathcal{L}_\epsilon(u)(x) = C_{N,1-s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2-2s}{2}}} dy.$$

Considering this approach to model (1.1) allows us to prove certain L^p -estimates. These are an essential tool in order to derive convergence of the solutions of approximating problems.

We start by assuming initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and we prove existence of a class of weak solutions that we construct using an approximating method. The paper combines a great variety of compactness techniques and the detailed proofs show how the available energy estimates can be used step by step as we pass to the limit in the approximating models. The main difficulties of the construction are: the nonlocal and nonlinear character of the equation, absence of comparison principle, absence of explicit self-similar solutions (except very particular cases, c.f [31]).

A second contribution of the paper is the generality of the initial data. We may take $u_0 = \mu \in \mathcal{M}^+(\mathbb{R}^N)$, the space of nonnegative Radon measures on \mathbb{R}^N with finite mass. This covers in particular the case of merely integrable data $u_0 \in L^1(\mathbb{R}^N)$. We cover that issue in Section 5 where we obtain existence of weak solutions for the whole range $1 < m < \infty$, generalizing the results of [9] and [32], where the cases $m = 2$ and $m \in (1, 3)$ were covered respectively. This rounds up the existence theory.

Another positive property of this approach is that it can be successfully generalized to more general equations of the form

$$u_t(x, t) = \nabla \cdot (G'(u) \nabla (-\Delta)^{-s} u),$$

where $G : [0, +\infty) \rightarrow [0, +\infty)$ is a regular function with at most linear growth at the origin.

A remarkable property of many diffusive PDE's of degenerate type is the finite speed of propagation. When we combine degenerate nonlinearities (powers with $m > 1$) and nonlocal effects it is not clear whether finite propagation will hold or not. The property was first observed by Caffarelli and Vázquez in [9] for the model with $m = 2$, see also [3] for $N = 1$. In [32] we discovered that the nonlinearity has a strong influence on the speed of propagation property of solutions independently of $s \in (0, 1)$. Indeed, we proved two different behaviors depending on the exponent m : finite speed of propagation for $m \in (2, 3)$ and infinite speed of propagation for $m \in (1, 2)$. A numerical simulation using [12] pointed us to this change in the positivity property of the solution. We establish here the property of finite propagation for all $m \geq 2$. See Figure 2.

Let us comment on some related literature. Another possible generalization of the model studied by Caffarelli and Vázquez in [9] has been considered in [2, 3, 20]. They assume that $p = (-\Delta)^{-s} u^{m-1}$. In this case, there exists a weak solution with finite speed of propagation for the whole range $m > 1$. Moreover, they find explicit Barenblatt self-similar profiles.

We finally recall that there is another model of nonlocal porous medium equation:

$$(1.2) \quad v_t + (-\Delta)^s (v^m) = 0$$

with $m > 0$ and $s \in (0, 1)$ for which the theory has been developed in [23, 24, 5, 36], see also the survey paper [35]. Infinite propagation holds for this model even if $m > 1$. A very interesting result is the connection between model (1.1) and model (1.2). In [31] we found an exact transformation formula between self-similar solutions of the two models, but it only applies to the range $m < 2$ of our present model.

2 Precise statement of the main results

We recall that all data and solutions are nonnegative and we will stress this fact when convenient. In this section will only present the results for integrable and bounded initial data since establishing the existence and main properties in this case contains the main difficulties. For clarity of exposition, we delay to Section 5 the case of measure data since it is an independent contribution of the paper.

Definition 2.1. Let $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ and nonnegative. We say that $u \geq 0$ is a weak solution of Problem (1.1) if:

(i) $u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$, (ii) $\nabla(-\Delta)^{-s}u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$, (iii) $u^{m-1}\nabla(-\Delta)^{-s}u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$ and

$$\int_0^T \int_{\mathbb{R}^N} u \phi_t dx dt - \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla(-\Delta)^{-s}u \cdot \nabla \phi dx dt + \int_{\mathbb{R}^N} u_0(x) \phi(x, 0) dx = 0$$

for all test functions $\phi \in C^1_c(\mathbb{R}^N \times [0, T])$.

We state our main results on the existence and qualitative properties of solutions.

Theorem 2.2. Let $1 < m < \infty$, $N \geq 1$, and let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and nonnegative. Then there exists a weak solution $u \geq 0$ of Problem (1.1) such that $u \in L^1(\mathbb{R}^N \times (0, T))$, $u \in L^\infty(\mathbb{R}^N \times (0, T))$, and $(-\Delta)^{\frac{1-s}{2}}u^r \in L^2(\mathbb{R}^N \times (0, T))$ for all $r > m/2$. Moreover, u has the following properties:

1. (**Conservation of mass**) For all $0 < t < T$ we have $\int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx$.
2. (**L^∞ estimate**) For all $0 < t < T$ we have $\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty$.
3. (**L^p energy estimate**) For all $1 < p < \infty$ and $0 < t < T$ we have

$$(2.1) \quad \int_{\mathbb{R}^N} u^p(x, t) dx + \frac{4p(p-1)}{(m+p-1)^2} \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-1}{2}} \right|^2 dx dt \leq \int_{\mathbb{R}^N} u_0^p(x) dx.$$

4. (**Second energy estimate**) For all $0 < t < T$ we have

$$(2.2) \quad \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(t) \right|^2 dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} \left| \nabla(-\Delta)^{-s} u(t) \right|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 dx.$$

Remark 1. (a) The a priori estimates 1, 2, 3 and 4 for Problem (1.1) can be derived in a formal way as in Section 3 of [32]. A rigorous proof for 1, 2 and 4 when $m \in (1, 3)$ can be found in that paper. The approximation used there does not allow to cover the whole range $m \in (1, +\infty)$ because of the lack of an L^p type energy estimate like (2.1). However, 1 and 2 follow as in [32] and therefore they will not be discussed here.

(b) We would like to note that estimates (2.1) and (2.2) do not present any special form or extra difficulty when $m = 2$, $m = 3$ or $m > 3$, as it happened with the First Energy Estimate (6.1) used in [32] and [9]. See Section 6 for a more detailed discussion about this fact.

Theorem 2.3 (Smoothing effect). Let $u \geq 0$ be a weak solution of Problem (1.1) with nonnegative initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ as constructed in Theorem 2.2. Then,

$$(2.3) \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m,p} t^{-\gamma_p} \|u_0\|_{L^p(\mathbb{R}^N)}^{\delta_p} \quad \text{for all } t > 0,$$

where $\gamma_p = \frac{N}{(m-1)N+2p(1-s)}$, $\delta_p = \frac{2p(1-s)}{(m-1)N+2p(1-s)}$.

Proof. We combine (2.1) with the Nash-Gagliardo-Nirenberg Inequality (7.2) applied to the function $f = u^{(m+p-1)/2}$ to get a starting point for a Moser iteration. Then we continue as in Theorem 8.2 of [24] where the authors consider the model $u_t + (-\Delta)^{\sigma/2} u^m = 0$ for $\sigma = 2 - 2s$. From here, the proof is straightforward. \square

Remark 2. In the limit $m \rightarrow 1^+$, Theorems 2.2, 2.3 (and also Theorem 5.2) recover some of the results of the linear Fractional Heat Equation (cf. [4]).

Theorem 2.4. *Let $m \geq 2$, $N \geq 1$, $s \in (0, 1)$. Let u be a weak solution of Problem (1.1) as constructed in Theorem 5.2 with compactly supported initial data $u_0 \in L^1(\mathbb{R}^N)$. Then $u(\cdot, t)$ is compactly supported for all $t > 0$, i.e. the solution has finite speed of propagation.*

Proof. Once we construct a weak solution of Problem (1.1), we apply the results from [32]. The proof is based on a careful construction of barrier functions, called *true supersolutions* in [9]. \square

3 Functional setting

3.1 The fractional Laplacian and the inverse operator

We remind some definitions and basic notions for the functional setting of the problem. We will work with the following functional spaces (see [16]). Let \mathcal{F} denote the Fourier transform. For given $s \in (0, 1)$ we consider the space

$$H^s(\mathbb{R}^N) := \left\{ u : L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\},$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi.$$

For functions $u \in H^s(\mathbb{R}^N)$, the fractional Laplacian operator is defined by

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)),$$

for $x \in \mathbb{R}^N$, where $C_{N,s} = \pi^{-(2s+N/2)} \Gamma(N/2 + s) / \Gamma(-s)$. Then,

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + C \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}.$$

For functions u that are defined on a subset $\Omega \subset \mathbb{R}^N$ with $u = 0$ on the boundary $\partial\Omega$, we will use the *restricted* version of the fractional Laplacian computed by extending the function u to the whole \mathbb{R}^N with $u = 0$ in $\mathbb{R}^N \setminus \Omega$. The same idea is used to define the $H^s(\mathbb{R}^N)$ norm for functions defined in Ω .

If $N > 2s$, the inverse operator $(-\Delta)^{-s}$ coincides with the Riesz potential of order $2s$. It can be represented by convolution with the Riesz kernel K_s :

$$(-\Delta)^{-s} u = K_s * u, \quad K_s(x) = \frac{1}{c(N, s)} |x|^{-(N-2s)},$$

where $c(N, s) = \pi^{N/2-2s} \Gamma(s) / \Gamma((N-2s)/2)$. Notice that $K_s \in L^1_{\text{loc}}(\mathbb{R}^N)$. When $N = 1$ and $s \in [1/2, 1)$ we have to consider the composed operator $\nabla(-\Delta)^{-s}$. This operator use to be called *nonlocal gradient* and is denoted by ∇^{1-2s} (c.f. [2, 32]). See Section 4.6 for a more detailed discussion of this range.

3.2 Approximation of the fractional Laplacian $(-\Delta)^s$

Let $\epsilon > 0$ and $u : \mathbb{R}^N \rightarrow \mathbb{R}$. We define the operator

$$(3.1) \quad \mathcal{L}_\epsilon^s[u](x) := C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dy,$$

for $x \in \mathbb{R}^N$. We will use the notation

$$J_\epsilon^s(z) := \frac{C_{N,s}}{(|z|^2 + \epsilon^2)^{\frac{N+2s}{2}}} \quad \text{for } z \in \mathbb{R}^N.$$

This kind of zero-order operators has been considered in the literature, see e. g. [19, 27]. For any $\epsilon > 0$, \mathcal{L}_ϵ^s is an integral operator with non-singular kernel and $\mathcal{L}_\epsilon[u] \rightarrow (-\Delta)^s u$ pointwise in \mathbb{R}^N as $\epsilon \rightarrow 0$ for suitable functions u . This approximation can also be seen as a consequence of the fact that the fractional Laplacian can be computed by passing to the limit in the representation of the solution of an harmonic extension problem (using the explicit Poisson formula), as proved by Caffarelli and Silvestre in [6].

We can define the bilinear form

$$\mathcal{E}_\epsilon(u, v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy \quad \text{for } u, v \in D(\mathcal{L}_\epsilon),$$

and the quadratic form

$$\bar{\mathcal{E}}_\epsilon(u) := \mathcal{E}_\epsilon(u, u) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)]^2}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy.$$

The bilinear form \mathcal{E}_ϵ is well defined for functions in the space $\dot{H}_\epsilon^s(\mathbb{R}^N)$, which is the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm given by $\bar{\mathcal{E}}_\epsilon$. We define

$$(3.2) \quad H_\epsilon^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \bar{\mathcal{E}}_\epsilon(u) < \infty\}.$$

The space $H_\epsilon^s(\mathbb{R}^N)$ is endowed with the standard norm

$$\|u\|_{H_\epsilon^s}^2 = \|u\|_{L^2(\mathbb{R}^N)}^2 + \bar{\mathcal{E}}_\epsilon(u).$$

Clearly,

$$(3.3) \quad H^s(\mathbb{R}^N) \subset H_{\epsilon_1}^s(\mathbb{R}^N) \subset H_{\epsilon_2}^s(\mathbb{R}^N) \quad \text{for } 0 < \epsilon_1 < \epsilon_2.$$

We refer to [15] for a precise discussion of these spaces in a more general framework.

Lemma 3.1. *Let $0 < s < 1$. Then, for every $\epsilon > 0$, we have that*

$$\mathcal{L}_\epsilon^s : L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \rightarrow L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Moreover,

$$\begin{aligned} \|\mathcal{L}_\epsilon^s[u]\|_{L^1(\mathbb{R}^N)} &\leq 2\|u\|_{L^1(\mathbb{R}^N)}\|J_\epsilon^s\|_{L^1(\mathbb{R}^N)}, \\ \|\mathcal{L}_\epsilon^s[u]\|_{L^\infty(\mathbb{R}^N)} &\leq 2\|u\|_{L^\infty(\mathbb{R}^N)}\|J_\epsilon^s\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Proof. It is clear that $\|J_\epsilon^s\|_{L^1(\mathbb{R}^N)} < \infty$ since J_ϵ^s is integrable at infinity and nonsingular at the origin. Let $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then

$$\begin{aligned}\|\mathcal{L}_\epsilon^s[u]\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} (u(x) - u(x+y)) J_\epsilon^s(y) dy \right| dx \\ &\leq \int_{\mathbb{R}^N} J_\epsilon^s(y) \int_{\mathbb{R}^N} |u(x) - u(x+y)| dx dy \\ &\leq 2\|u\|_{L^1(\mathbb{R}^N)} \int_{\mathbb{R}^N} J_\epsilon^s(y) dy,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{L}_\epsilon^s[u](x)| &= \left| \int_{\mathbb{R}^N} (u(x) - u(x+y)) J_\epsilon^s(y) dy \right| \\ &\leq \int_{\mathbb{R}^N} |u(x) - u(x+y)| J_\epsilon^s(y) dy \\ &\leq 2\|u\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} J_\epsilon^s(y) dy.\end{aligned}$$

□

The restricted operator. For smooth functions $f : B_R \rightarrow \mathbb{R}$ such that $f = 0$ on ∂B_R we extend $f := 0$ on $\mathbb{R}^N \setminus B_R$. Then \mathcal{L}_ϵ^s is well defined by definition (3.1). Let $C_c^\infty(B_R) = \{f \in C_c^\infty(\mathbb{R}^N) \text{ with } \text{supp}(f) \subset B_R\}$. We take $\dot{H}_\epsilon^s(B_R)$ to be the closure of $C_c^\infty(B_R)$ with respect to the quadratic form $\bar{\mathcal{E}}_\epsilon$. Then \mathcal{L}_ϵ^s is well defined on $\dot{H}_\epsilon^s(B_R)$.

Square root. The operator \mathcal{L}_ϵ^s has a square-root in the Fourier transform sense (see [14] Lemma 3.7), that we denote by $(\mathcal{L}_\epsilon^s)^{\frac{1}{2}}$. We have that

$$\langle u, \mathcal{L}_\epsilon^s[u] \rangle_{L^2(\mathbb{R}^N)} = \|(\mathcal{L}_\epsilon^s)^{\frac{1}{2}}[u]\|_{L^2(\mathbb{R}^N)}^2.$$

This implies that

$$\begin{aligned}\langle \mathcal{L}_\epsilon^s[u], u \rangle_{L^2(\mathbb{R}^N)} &= C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(x) \frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy \\ &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)]^2}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy \\ &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[\frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{4}}} \right]^2 dx dy,\end{aligned}$$

where the second identity is obtained by symmetry. We get the following characterization of $(\mathcal{L}_\epsilon^s)^{\frac{1}{2}}$:

$$(3.4) \quad \int_{\mathbb{R}^N} \left((\mathcal{L}_\epsilon^s)^{\frac{1}{2}}[u](x) \right)^2 dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[\frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{4}}} \right]^2 dx dy.$$

Theorem 3.2 (Generalized Stroock-Varopoulos Inequality for \mathcal{L}_ϵ^s). Let $u \in H_\epsilon^s(\mathbb{R}^N)$. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi \in C^1(\mathbb{R})$ and $\psi' \geq 0$. Then

$$(3.5) \quad \int_{\mathbb{R}^N} \psi(u) \mathcal{L}_\epsilon^s[u] dx \geq \int_{\mathbb{R}^N} \left| (\mathcal{L}_\epsilon^s)^{\frac{1}{2}} [\Psi(u)] \right|^2 dx,$$

where $\psi' = (\Psi')^2$.

Proof. We have that:

$$\begin{aligned} \int_{\mathbb{R}^N} \psi(u) \mathcal{L}_\epsilon^s[u] dx &= C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi(u(x)) \frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy \\ &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [\psi(u(x)) - \psi(u(y))] \frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy. \end{aligned}$$

Now, we use that if ψ is such that $\psi' \geq 0$ and $\psi' = (\Psi')^2$, then

$$(\psi(a) - \psi(b))(a - b) \geq (\Psi(a) - \Psi(b))^2, \quad \forall a, b \in \mathbb{R}^N.$$

For convenience, we give the proof of this pointwise inequality based on the Fundamental Theorem of Calculus and the Cauchy-Schwarz Inequality:

$$\begin{aligned} (\Psi(a) - \Psi(b))^2 &= \left(\int_b^a \Psi'(z) dz \right)^2 \leq (a - b) \int_b^a (\Psi'(z))^2 dz \\ &= (a - b) \int_b^a \psi'(z) dz = (a - b)(\psi(a) - \psi(b)). \end{aligned}$$

We deduce, using (3.4), that

$$\int_{\mathbb{R}^N} \psi(u) \mathcal{L}_\epsilon^s[u] dx \geq \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[\Psi(u(x)) - \Psi(u(y))]^2}{(|x - y|^2 + \epsilon^2)^{\frac{N+2s}{2}}} dx dy = \int_{\mathbb{R}^N} \left| (\mathcal{L}_\epsilon^s)^{\frac{1}{2}} \Psi(u(x)) \right|^2 dx.$$

□

Remark 3. (i) We refer to [15] for a related result with more general nonlinearities and nonlocal operators.

(ii) Note that we recover the classical Stroock-Varopoulos Inequality for \mathcal{L}_ϵ by taking $\psi(u) = |u|^{q-2}u$:

$$\int_{\mathbb{R}^N} |u|^{q-2} u \mathcal{L}_\epsilon^s[u] dx \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^N} \left| (\mathcal{L}_\epsilon^s)^{1/2} (u^{q/2}) \right|^2 dx.$$

We refer to Stroock [34], Liskevich [21] where this kind of inequality is proved for general sub-markovian operators.

3.3 Approximation of the inverse fractional Laplacian $(-\Delta)^{-s}$, $s \in (0, 1)$

As a consequence of (3.1) we naturally derive an approximation for the inverse fractional Laplacian $(-\Delta)^{-s}$ and the nonlocal gradient ∇^{1-2s} that will play an important role in the sequel to solve the difficulties created by estimates like (6.1) in the range $m \geq 3$.

Lemma 3.3. a) Let $N \geq 1$, $s \in (0, 1)$ and $s < \frac{N}{2}$. Then for every $f \in L^1(\mathbb{R}^N)$ such that $(-\Delta)^{-s}f \in L^2(\mathbb{R}^N)$ we have that

$$\mathcal{I}_\epsilon := \int_{\mathbb{R}^N} ((-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[f] - (-\Delta)^{-s}f) \phi dx \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^N).$$

b) Let $N \geq 1$, $s \in (0, 1)$. Then for every $f \in L^1(\mathbb{R}^N)$ such that $\nabla^{1-2s}f \in L^2(\mathbb{R}^N)$ we have that

$$\mathcal{I}_\epsilon := \int_{\mathbb{R}^N} (\nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[f] - \nabla^{1-2s}f) \phi dx \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^N).$$

Proof. a) Given any operator T , let $S_T(\xi)$ be the Fourier symbol associated to the operator T whenever it is well defined. Now, we employ Plancherel's Theorem to obtain:

$$\mathcal{I}_\epsilon = \int_{\mathbb{R}^N} \left(\mathcal{S}_{(-\Delta)^{-1}}(\xi) \mathcal{S}_{\mathcal{L}_\epsilon^{1-s}}(\xi) - \mathcal{S}_{(-\Delta)^{-s}}(\xi) \right) \widehat{f} \widehat{\phi} d\xi =: \int_{\mathbb{R}^N} F_\epsilon(\xi) d\xi.$$

We want to pass to the limit as $\epsilon \rightarrow 0$ in \mathcal{I}_ϵ . For that purpose we need to find an L^1 dominating function for F_ϵ . We recall that for $s \in (0, 1)$ we have that

$$(3.6) \quad \mathcal{S}_{\mathcal{L}_\epsilon^{1-s}}(\xi) = \int_{|z|>0} \frac{1 - \cos(z \cdot \xi)}{(|z|^2 + \epsilon^2)^{\frac{N+2(1-s)}{2}}} dz \quad \text{and} \quad \mathcal{S}_{(-\Delta)^{1-s}}(\xi) = \int_{|z|>0} \frac{1 - \cos(z \cdot \xi)}{|z|^{N+2(1-s)}} dz \sim |\xi|^{2(1-s)}.$$

Moreover $\mathcal{S}_{(-\Delta)^{-s}}(\xi) = \mathcal{S}_{(-\Delta)^{-1}}(\xi) \mathcal{S}_{(-\Delta)^{1-s}}(\xi)$. Note that $0 \leq \mathcal{S}_{\mathcal{L}_\epsilon^{1-s}}(\xi) \leq \mathcal{S}_{(-\Delta)^{1-s}}(\xi)$ for every $\xi \in \mathbb{R}^N$. Then

$$\begin{aligned} |F_\epsilon(\xi)| &\leq \left| \mathcal{S}_{(-\Delta)^{-1}}(\xi) \mathcal{S}_{\mathcal{L}_\epsilon^{1-s}}(\xi) \right| \left| \widehat{f} \right| \left| \widehat{\phi} \right| + \left| \mathcal{S}_{(-\Delta)^{-s}}(\xi) \right| \left| \widehat{f} \right| \left| \widehat{\phi} \right| \\ &\leq \left| \mathcal{S}_{(-\Delta)^{-1}}(\xi) \mathcal{S}_{(-\Delta)^{1-s}}(\xi) \right| \left| \widehat{f} \right| \left| \widehat{\phi} \right| + \left| \mathcal{S}_{(-\Delta)^{-s}}(\xi) \right| \left| \widehat{f} \right| \left| \widehat{\phi} \right| \\ &= 2 \left| \mathcal{S}_{(-\Delta)^{-s}}(\xi) \widehat{f} \widehat{\phi} \right| \leq C |\xi|^{-2s} \left| \widehat{f} \right| \left| \widehat{\phi} \right|. \end{aligned}$$

We conclude that $|F_\epsilon(\xi)| \leq G(\xi, t) := C \left| |\xi|^{-2s} \widehat{f} \widehat{\phi} \right| \in L^1(\mathbb{R}^N)$ since $\widehat{f} \in L^\infty(\mathbb{R}^N)$ and $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^N)$, the Schwartz space of rapidly decaying functions. Moreover, we can see from (3.6) that $F_\epsilon(\xi) \rightarrow 0$ pointwise as $\epsilon \rightarrow 0$. Then we use the Dominated Convergence Theorem to conclude that $|\mathcal{I}_\epsilon| \rightarrow 0$ as $\epsilon \rightarrow 0$.

b) The proof follows as above noting that $\mathcal{S}_\nabla = i\xi$ and $|F_\epsilon(\xi)| \leq C \left| |\xi|^{1-2s} \widehat{f} \widehat{\phi} \right| \in L^1(\mathbb{R}^N)$.

□

4 Existence of weak solutions via approximating problems

In order to prove existence of weak solutions of Problem (1.1) we proceed by considering an approximating problem. We regularize the degeneracy of the nonlinearity, the singularity of the fractional operator, we also add a vanishing viscosity term to get more regularity and we restrict the problem to a bounded domain. We write the equation in the form

$$u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-1} (-\Delta)^{1-s} u).$$

The idea is to consider the approximation of the $(-\Delta)^{1-s}$ given by (3.1), that is

$$\mathcal{L}_\epsilon^{1-s}(u)(x) = C_{N,1-s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2-2s}{2}}} dy,$$

defined for functions u in the space $H_\epsilon^s(\mathbb{R}^N)$. We consider the approximating problem

$$(P_{\epsilon\delta\mu R}) \quad \begin{cases} (U_1)_t = \delta\Delta U_1 + \nabla \cdot ((U_1 + \mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1]) & \text{for } (x, t) \in B_R \times (0, T), \\ U_1(x, 0) = \hat{u}_0(x) & \text{for } x \in B_R, \\ U_1(x, t) = 0 & \text{for } x \in \partial B_R, t \in (0, T), \end{cases}$$

with parameters $\epsilon, \delta, \mu, R > 0$. We use the notation $B_R := B_R(0)$. The initial data \hat{u}_0 is a smooth approximation of u_0 .

Definition 4.1. We say that U_1 is a weak solution of Problem $(P_{\epsilon\delta\mu R})$ if: (i) $U_1 \in L^1(B_R \times (0, T))$, (ii) $\nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \in L^1(B_R \times (0, T))$, (iii) $(U_1 + \mu)^{m-1} \nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \in L^1(B_R \times (0, T))$ and

$$(4.1) \quad \int_0^T \int_{B_R} U_1(\phi_t - \delta\Delta\phi) dx dt - \int_0^T \int_{B_R} (U_1 + \mu)^{m-1} \nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \cdot \nabla\phi dx dt + \int_{B_R} \hat{u}_0(x)\phi(x, 0) dx = 0$$

for smooth test functions ϕ that vanish on the spatial boundary ∂B_R and for large t .

An important tool in the proof of existence of weak solutions is the concept of mild solution of Problem $(P_{\epsilon\delta\mu R})$, i.e. fixed points of the following map given by the Duhamel's formula

$$\mathcal{T}(v)(x, t) = e^{\delta t \Delta} u_0(x) + \int_0^t \nabla e^{\delta(t-\tau)\Delta} \cdot G(v)(x, \tau) d\tau, \quad G(v) = (v + \mu)^{m-1} \nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^s[v],$$

where $e^{t\Delta}$ is the Heat Semigroup. The map $v \mapsto \mathcal{T}(v)$,

$$\mathcal{T} : C((0, T) : L^1(B_R) \cap L^\infty(B_R)) \rightarrow C((0, T) : L^1(B_R) \cap L^\infty(B_R))$$

is well defined and moreover, \mathcal{T} is a contraction. By the Banach contraction principle we obtain that there exists a fixed point $\mathcal{T}(U_1) = U_1$. It remains to prove that U_1 is a weak solution of Problem $(P_{\epsilon\delta\mu R})$. The method of mild solutions via Duhamel formula was successfully employed to prove existence of approximated solutions for another nonlocal porous medium model by Biler, Imbert and Karch in [2]. Their approximation is slightly different but the technical part can be adapted to Problem $(P_{\epsilon\delta\mu R})$. In fact one can prove that the solutions are classical.

Notations. • The existence of a weak solution of Problem (1.1) is done by passing to the limit step-by-step in the approximating problems as follows. We denote by U_1 the solution of the approximating Problem $(P_{\epsilon\delta\mu R})$ with parameters ϵ, δ, μ, R . Afterwards, we obtain $U_2 = \lim_{\epsilon \rightarrow 0} U_1$ and U_2 solves an approximating Problem $(P_{\delta\mu R})$ with parameters δ, μ, R . Next, we take $U_3 = \lim_{R \rightarrow \infty} U_2$ that will be a solution of Problem $(P_{\delta\mu})$, $U_4 := \lim_{\mu \rightarrow 0} U_3$ solving Problem (P_δ) . Finally we obtain $u = \lim_{\delta \rightarrow 0} U_4$ which solves Problem (1.1).

• We will often use $\int_0^t f(t) dt$ to avoid introducing new variables. Also, we will use $\int_{\mathbb{R}^N}$ instead of \int_{B_R} when integrating some expressions of U_1, U_2 , which are supported in B_R , by identifying these functions with 0 outside the domain B_R . The homogeneous Dirichlet boundary conditions ensures that the integrals coincide.

• We will use \rightarrow for strong convergence and \rightharpoonup for weak convergence.

4.1 A-priori estimates for the approximating problem ($P_{\epsilon\delta\mu R}$)

Let U_1 be a smooth solution of Problem ($P_{\epsilon\delta\mu R}$). Then we have the following a-priori estimates.

- **L^p energy estimates for $1 \leq p < \infty$.** For all $0 < t < T$ we have that:

$$\begin{aligned} \frac{d}{dt} \int_{B_R} U_1^p(x, t) dx &= p \int_{B_R} U_1^{p-1} (U_1)_t dx = \\ &= -p\delta \int_{B_R} \nabla(U_1^{p-1}) \cdot \nabla U_1 dx - p \int_{B_R} \nabla U_1^{p-1} (U_1 + \mu)^{m-1} \cdot \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] dx \\ &= -\frac{4(p-1)\delta}{p} \int_{B_R} \left| \nabla(U_1^{p/2}) \right|^2 dx - p(p-1) \int_{B_R} U_1^{p-2} (U_1 + \mu)^{m-1} \nabla u \cdot \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] dx. \end{aligned}$$

The boundary terms are 0 since $U_1 = 0$ on ∂B_R . We analyze the second term:

$$\begin{aligned} \int_{B_R} U_1^{p-2} (U_1 + \mu)^{m-1} \nabla U_1 \cdot \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] dx &= \int_{B_R} \nabla \psi(U_1) \cdot \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] dx \\ &= \int_{B_R} \psi(U_1) (-\Delta) (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] dx = \int_{B_R} \psi(U_1) \mathcal{L}_\epsilon^{1-s} [U_1] dx \\ &\geq \int_{B_R} |(\mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [\Psi(U_1)]|^2 dx. \end{aligned}$$

We have used the generalized Stroock-Varopoulos Inequality (3.5) in the following context: the functions ψ and Ψ are such that $\psi' = (\Psi')^2$ and $\nabla \psi(U_1) = U_1^{p-2} (U_1 + \mu)^{m-1} \nabla U_1$. The precise definition of these functions is given by

$$\psi(z) = \int_0^z \zeta^{p-2} (\zeta + \mu)^{m-1} d\zeta, \quad \Psi(z) = \int_0^z \zeta^{\frac{p-2}{2}} (\zeta + \mu)^{\frac{m-1}{2}} d\zeta.$$

We obtain the following L^p -energy estimate:

$$(4.2) \quad \int_{B_R} u_0^p(x) dx - \int_{B_R} U_1^p(x, t) dx = \frac{4(p-1)\delta}{p} \int_0^t \int_{B_R} \left| \nabla(U_1^{p/2}) \right|^2 dx dt + p(p-1) \int_0^t \int_{B_R} \psi(u) \mathcal{L}_\epsilon^{1-s} [U_1] dx dt,$$

and then

$$\begin{aligned} \int_{B_R} u_0^p(x) dx &\geq \int_{B_R} U_1^p(x, t) dx + \\ &+ \frac{4(p-1)\delta}{p} \int_0^t \int_{B_R} \left| \nabla(U_1^{p/2}) \right|^2 dx dt + p(p-1) \int_0^t \int_{B_R} |(\mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [\Psi(U_1)]|^2 dx dt. \end{aligned}$$

As a consequence we get that $\Psi(U_1) \in L^2(0, T : H_\epsilon^{1-s}(\mathbb{R}^N))$ for $u_0 \in L^p(\mathbb{R}^N)$.

• **Second Energy Estimate**

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{B_R} |((-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [U_1]|^2 dx = \int_{B_R} ((-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [U_1] \cdot ((-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [(U_1)_t] dx \\
& = \int_{B_R} (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] (U_1)_t dx \\
& = \delta \int_{B_R} (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] \Delta U_1 dx + \int_{B_R} (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1] \nabla \cdot ((U_1 + \mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1]) dx \\
& = -\delta \int_{B_R} |(\mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [U_1]|^2 dx - \int_{B_R} (U_1 + \mu)^{m-1} |\nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1]|^2 dx.
\end{aligned}$$

Therefore, the quantity $\int_{B_R} |(-\Delta)^{-\frac{1}{2}} (\mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [U_1](x, t)|^2 dx$ is non-increasing in t and we have that

$$\begin{aligned}
(4.3) \quad & \frac{1}{2} \int_{B_R} |((-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [u_0]|^2 dx = \frac{1}{2} \int_{B_R} |((-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [U_1(t)]|^2 dx \\
& + \delta \int_0^t \int_{B_R} |(\mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}} [U_1]|^2 dx dt + \int_0^t \int_{B_R} (U_1 + \mu)^{m-1} |\nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s} [U_1]|^2 dx dt.
\end{aligned}$$

We conclude the results obtained so far in the following theorem.

Theorem 4.2. *Let $s \in (0, 1)$, $1 < m < \infty$ and $N \geq 1$. There exists a weak solution U_1 of Problem $(P_{\epsilon\delta\mu R})$ with initial data \widehat{u}_0 . Moreover U_1 satisfies the L^p -energy estimate (4.2), the second energy estimate (4.3) and also*

1. (**Decay of total mass**) For all $0 < t < T$ we have $\int_{B_R} U_1(x, t) dx \leq \int_{B_R} u_0(x) dx$.
2. (**L^∞ -estimate**) For all $0 < t < T$ we have $\|U_1(\cdot, t)\|_\infty \leq \|u_0\|_\infty$.

Remark 4. In Sections 4.2, 4.3, 4.4 and 4.5 we will only consider $s \in (0, \frac{1}{2})$ when $N = 1$ since the operator $(-\Delta)^{-s}$ is not well defined out of this range. We will devote Section 4.6 to comment how to deal with the case $N = 1$, $s \in [\frac{1}{2}, 1)$.

4.2 Limit as $\epsilon \rightarrow 0$

Let U_1 be a weak solution of problem $(P_{\epsilon\delta\mu R})$ with parameters $\delta, \mu, R > 0$ fixed from the beginning. We will prove that $\lim_{\epsilon \rightarrow 0} U_1 = U_2$, where U_2 is a weak solution of the problem

$$(P_{\delta\mu R}) \quad \begin{cases} (U_2)_t = \delta \Delta U_2 + \nabla \cdot ((U_2 + \mu)^{m-1} \nabla (-\Delta)^{-s} U_2) & \text{for } (x, t) \in B_R \times (0, T), \\ U_2(x, 0) = \widehat{u}_0(x) & \text{for } x \in B_R, \\ U_2(x, t) = 0 & \text{for } x \in \partial B_R, t \geq 0. \end{cases}$$

Moreover, we will also prove that U_2 inherits most of the properties of U_1 .

4.2.1 Existence of a limit. Compactness estimate I

I. Using the energy estimate (4.2) with $p = 2$ we obtain that $U_1 \in L^2(0, T : H_0^1(B_R))$.

II. The second energy estimate (4.3) gives us that $(\mathcal{L}_\epsilon^{1-s})^{\frac{1}{2}}[U_1] \in L^2(0, T : L^2(B_R))$. Therefore $U_1 \in L^2(0, T : H_\epsilon^{1-s}(B_R))$ where $H_\epsilon^{1-s}(B_R)$ is defined as in (3.2). Consequently, for any $0 < \epsilon < \epsilon_0 \leq 1$, we have

$$U_1 \in L^2(0, T : H_{\epsilon_0}^{1-s}(B_R))$$

uniformly bounded on ϵ .

III. Estimates on the derivative $(U_1)_t$. We use the equation

$$(U_1)_t = \delta \Delta U_1 + \nabla \cdot ((U_1 + \mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1]).$$

The H_0^1 estimate of (4.2) ensures that $\delta \Delta U_1 \in L^2(0, T : H^{-1}(B_R))$. The second energy estimate (4.3) implies that

$$(U_1 + \mu)^{\frac{m-1}{2}} \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \in L^2(0, T : L^2(B_R)).$$

Since also $U_1 \in L^\infty((0, T) \times B_R)$ then this implies that $\nabla \cdot ((U_1 + \mu)^{m-1} \nabla (-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1]) \in L^2(0, T : H^{-1}(B_R))$. We conclude that

$$(U_1)_t \in L^2(0, T : H^{-1}(B_R)).$$

IV. We apply the compactness criteria of Simon (see Lemma 7.5 in the Section 7) in the context of

$$(4.4) \quad H_{\epsilon_0}^{1-s}(B_R) \subset L^2(B_R) \subset H^{-1}(B_R),$$

where the left hand side inclusion is compact. We conclude that the family of approximate solutions $\{U_1\}_{\epsilon>0}$ is relatively compact in $L^2(0, T : L^2(B_R))$. Therefore, there exists a limit $(U_1)_{\epsilon, \delta, \mu, R} \rightarrow (U_2)_{\delta, \mu, R}$ as $\epsilon \rightarrow 0$ in $L^2(0, T : L^2(B_R))$, up to subsequences. Note that, since $(U_1)_\epsilon$ is a family of positive functions defined on B_R and extended to 0 in $\mathbb{R}^N \setminus B_R$, then the limit $U_2 = 0$ a.e. on $\mathbb{R}^N \setminus B_R$. We obtain that

$$(4.5) \quad U_1 \xrightarrow{\epsilon \rightarrow 0} U_2 \quad \text{in } L^2(0, T : L^2(B_R)) = L^2(B_R \times (0, T)).$$

4.2.2 The limit U_2 is a solution of the new problem ($P_{\delta\mu R}$)

We pass to the limit as $\epsilon \rightarrow 0$ in the definition (4.1) of a weak solution of Problem ($P_{\epsilon\delta\mu R}$) and we prove that the limit U_2 found in (4.5) is a weak solution of Problem ($P_{\delta\mu R}$). The convergence of the first integral in (4.1) is justified by (4.5) since

$$(4.6) \quad \left| \int_0^T \int_{B_R} (U_1 - U_2)(\phi_t - \delta \Delta \phi) dx dt \right| \leq \|U_1 - U_2\|_{L^2(B_R \times (0, T))} \|\phi_t - \delta \Delta \phi\|_{L^2(B_R \times (0, T))}.$$

To prove convergence of the second integral in (4.1) we argue as follows. Using (4.5) and the L^∞ -decay estimate from Theorem 4.2 we get that

$$(4.7) \quad (U_1 + \mu)^{m-1} \rightarrow (U_2 + \mu)^{m-1} \quad \text{in } L^2(B_R \times (0, T)).$$

The convergence of the nonlocal gradient term in (4.1) is proved in the following lemma.

Lemma 4.3. *We have that*

$$\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] \xrightarrow{\epsilon \rightarrow 0} \nabla(-\Delta)^{-s}U_2 \quad \text{in} \quad L^2(B_R \times (0, T)).$$

Proof. I. There exists a weak limit. From the second energy estimate (4.3) we note that

$$\begin{aligned} \|\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1]\|_{L^2(B_R \times (0, T))} &= \left\| \frac{(U_1 + \mu)^{\frac{m-1}{2}}}{(U_1 + \mu)^{\frac{m-1}{2}}} \nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] \right\|_{L^2(B_R \times (0, T))} \\ &\leq \mu^{-\frac{m-1}{2}} \left\| (U_1 + \mu)^{\frac{m-1}{2}} \nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] \right\|_{L^2(B_R \times (0, T))} \leq C. \end{aligned}$$

Then, Banach-Alaoglu Theorem ensures that there exists a subsequence such that

$$\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] \xrightarrow{\epsilon \rightarrow 0} v \text{ in } L^2(B_R \times (0, T)).$$

II. Identifying the limit in the sense of distributions. Now, we will prove that

$$\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] \xrightarrow{\epsilon \rightarrow 0} \nabla(-\Delta)^{-s}U_2$$

in distributions. More exactly, we will prove that

$$\int_0^T \int_{B_R} (-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] \nabla \phi dx dt \xrightarrow{\epsilon \rightarrow 0} \int_0^T \int_{B_R} (-\Delta)^{-s}U_2 \nabla \phi dx dt$$

for all $\phi \in C_c^\infty(B_R \times (0, T))$. We estimate the difference of the two integrals above as follows,

$$\begin{aligned} I_\epsilon &= \int_0^T \int_{B_R} ((-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1] - (-\Delta)^{-s}U_1) \nabla \phi dx dt + \int_0^T \int_{B_R} ((-\Delta)^{-s}U_1 - (-\Delta)^{-s}U_2) \nabla \phi dx dt \\ &= I_{1,\epsilon} + I_{2,\epsilon}. \end{aligned}$$

The first integral converges to 0 as a consequence of the approximation of $(-\Delta)^{-s}$ in the sense derived in Lemma 3.3 a). Note that U_1 is changing with ϵ , but we have the uniform bound $\|U_1\|_1 \leq \|u_0\|_1$ which ensures that Lemma 3.3 can still be applied. For the second integral we write

$$\begin{aligned} I_{2,\epsilon} &= \int_0^T \int_{\mathbb{R}^N} (U_1 - U_2) \nabla(-\Delta)^{-s} \phi dx dt \\ &= \int_0^T \int_{B_\rho} (U_1 - U_2) \nabla(-\Delta)^{-s} \phi dx dt + \int_0^T \int_{\mathbb{R}^N \setminus B_\rho} (U_1 - U_2) \nabla(-\Delta)^{-s} \phi dx dt \end{aligned}$$

for a ρ to be chosen later. Now fix $\eta > 0$. Then

$$\begin{aligned} (4.8) \quad \int_0^T \int_{\mathbb{R}^N \setminus B_\rho} |U_1 - U_2| |\nabla(-\Delta)^{-s} \phi| dx dt &\leq \|U_1 - U_2\|_{L^2(\mathbb{R}^N \times (0, T))} \|\nabla(-\Delta)^{-s} \phi\|_{L^2((\mathbb{R}^N \setminus B_\rho) \times (0, T))} \\ &\leq 2T \|u_0\|_{L^2(\mathbb{R}^N)} \|\nabla(-\Delta)^{-s} \phi\|_{L^2((\mathbb{R}^N \setminus B_\rho) \times (0, T))}. \end{aligned}$$

Since $\nabla(-\Delta)^{-s} \phi \in L^2(\mathbb{R}^N \times (0, T))$ then we can choose ρ large enough such that $\|\nabla(-\Delta)^{-s} \phi\|_{L^2((\mathbb{R}^N \setminus B_\rho) \times (0, T))} \leq \eta/2$. On the other hand

$$\int_0^T \int_{B_\rho} (U_1 - U_2) \nabla(-\Delta)^{-s} \phi dx dt \leq \|U_1 - U_2\|_{L^2(B_\rho \times (0, T))} \|\nabla(-\Delta)^{-s} \phi\|_{L^2(B_\rho \times (0, T))}.$$

We choose ϵ small enough such that $\|U_1 - U_2\|_{L^2(B_\rho \times (0, T))} \leq \eta/2$. Therefore $I_{2,\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Note that we could have fixed $\rho = R$ and then the first integral in (4.8) is identically zero since U_1 and U_2 are supported in B_R . We keep the splitting here since it will be needed to estimate $I_{2,\epsilon}$ in the limit as $R \rightarrow \infty$ (see Section 4.3.2). \square

To conclude this part, we use the following: given two sequences $f_\epsilon \rightharpoonup f$ in L^2 and $g_\epsilon \rightarrow g$ strongly in L^2 , then the scalar product converges $\int f_\epsilon g_\epsilon dx \rightarrow \int fg dx$. Then (4.7) together with Lemma 4.3 implies that

$$\int_0^T \int_{B_R} (U_1 + \mu)^{m-1} \nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \nabla \phi dx dt \xrightarrow{\epsilon \rightarrow 0} \int_0^T \int_{B_R} (U_2 + \mu)^{m-1} \nabla(-\Delta)^{-s} U_2 \nabla \phi dx dt.$$

4.2.3 Passing to the limit in the L^p energy estimate (4.2)

We have that

$$\begin{aligned} \int_{B_R} \psi(U_1) \mathcal{L}_\epsilon^{1-s}[U_1] dx &= \int_{B_R} \int_{B_R} \psi(U_1(x)) \frac{U_1(x) - U_1(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2(1-s)}{2}}} dx dy \\ &= \frac{1}{2} \int_{B_R} \int_{B_R} (\psi(U_1(x)) - \psi(U_1(y))) \frac{U_1(x) - U_1(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2(1-s)}{2}}} dx dy. \end{aligned}$$

Let

$$G_\epsilon(x, y) := \frac{1}{2} (\psi(U_1(x)) - \psi(U_1(y))) \frac{U_1(x) - U_1(y)}{(|x - y|^2 + \epsilon^2)^{\frac{N+2(1-s)}{2}}},$$

and

$$G(x, y) := \frac{1}{2} (\psi(U_2(x)) - \psi(U_2(y))) \frac{U_2(x) - U_2(y)}{|x - y|^{N+2(1-s)}}.$$

Note that $G_\epsilon(x, y) \geq 0$ since ψ is a non-decreasing function. Also, $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_\epsilon(x, y) \leq C$ uniformly in $\epsilon > 0$. Since $U_1 \rightarrow U_2$ as $\epsilon \rightarrow 0$ pointwise a.e. in $x \in B_R$ then $G_\epsilon(x, y) \rightarrow G(x, y)$ a.e. $x, y \in \mathbb{R}^N$. We can pass to the limit $\epsilon \rightarrow 0$ in the last term of the energy estimate (4.2) according to the Fatou's Lemma

$$(4.9) \quad \lim_{\epsilon \rightarrow 0} \int_0^t \int_{B_R} \int_{B_R} G_\epsilon(x, y) dx dy dt \geq \int_0^t \int_{B_R} \int_{B_R} G(x, y) dx dy = \int_0^t \int_{B_R} \psi(U_2) (-\Delta)^{1-s} U_2 dx dt.$$

Now we pass to the limit in the H^1 term. The L^p energy estimate (4.2) shows that $U_1^{p/2}$ is uniformly bounded in $L^2(0, T : H_0^1(B_R))$, therefore there exists a weak limit w in $L^2(0, T : H_0^1(B_R))$. Since $H_0^1(B_R) \subset L^2(B_R)$ with continuous inclusion, then $U_1^{p/2} \rightarrow w$ in $L^2(B_R \times (0, T))$. By (4.5) we know that $U_1 \rightarrow U_2$ in $L^2(B_R \times (0, T))$. For $p > 2$ we deduce that $U_1^{p/2} \rightarrow U_2^{p/2}$ in $L^2(B_R \times (0, T))$ and then we identify the limit $w = U_2^{p/2}$. The weak lower semi-continuity of the $\|\cdot\|_{H_0^1(B_R)}$ norm implies that

$$\liminf_{\epsilon \rightarrow 0} \int_0^t \int_{B_R} |\nabla(U_1^{p/2})|^2 dx dt \geq \int_0^t \int_{B_R} |\nabla(U_2^{p/2})|^2 dx dt.$$

We used the fact that the norm of a Hilbert space is weakly semi-continuous. A similar idea will be employed to pass to the limit also in the integrals in the second energy estimate (4.3).

4.2.4 Passing to the limit in the second energy estimate (4.3)

The first two terms involve integral operators, so the continuous inclusion $L^2(B_R) \subset H^{-s/2}(B_R)$ together with (4.5) allow to pass to the limit. For the third one we use the argument given in Section 4.2.3 in the particular case $\psi(U_1) = U_1$. For the last term we have to prove the following inequality

$$\liminf_{\epsilon \rightarrow 0} \int_0^t \int_{B_R} (U_1 + \mu)^{m-1} |\nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1]|^2 dx dt \geq \int_0^t \int_{B_R} (U_2 + \mu)^{m-1} |\nabla(-\Delta)^{-s} U_2|^2 dx dt.$$

This is a consequence of the fact that the L^2 norm is weakly lower semi-continuous and $(U_1 + \mu)^{\frac{m-1}{2}} \nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \rightharpoonup (U_2 + \mu)^{\frac{m-1}{2}} \nabla(-\Delta)^{-s} U_2$ in $L^2(B_R \times (0, t))$. Indeed, we have that

$$\int_0^t \int_{B_R} (U_1 + \mu)^{\frac{m-1}{2}} \nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \phi dx dt \xrightarrow{\epsilon \rightarrow 0} \int_0^t \int_{B_R} (U_2 + \mu)^{\frac{m-1}{2}} \nabla(-\Delta)^{-s} U_2 \phi dx dt$$

for every $\phi \in L^2(B_R \times (0, t))$. This is because $(U_1 + \mu)^{\frac{m-1}{2}} \phi \rightarrow (U_2 + \mu)^{\frac{m-1}{2}} \phi$ in $L^2(B_R \times (0, t))$ (using the Dominated Convergence Theorem) and $\nabla(-\Delta)^{-1} \mathcal{L}_\epsilon^{1-s}[U_1] \rightarrow \nabla(-\Delta)^{-s} U_2$ in $L^2(B_R \times (0, t))$ by Lemma 4.3.

From now on, we do not need to consider a smooth initial data $\widehat{u}_0 \sim u_0$. We sum up the results of this section in the following theorem.

Theorem 4.4. *Let $s \in (0, 1)$, $1 < m < \infty$, $N \geq 1$. There exists a weak solution U_2 of Problem $(P_{\delta\mu R})$ with initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover U_2 has the following properties:*

1. **(Decay of total mass)** For all $0 < t < T$ we have $\int_{B_R} U_2(x, t) dx \leq \int_{B_R} u_0(x) dx$.
2. **(L^∞ estimate)** For all $0 < t < T$ we have $\|U_2(\cdot, t)\|_\infty \leq \|u_0\|_\infty$.
3. **(L^p energy estimate)** For all $1 < p < \infty$ and $0 < t < T$ we have

$$(4.10) \quad \begin{aligned} \int_{B_R} U_2^p(x, t) dx + \frac{4(p-1)\delta}{p} \int_0^t \int_{B_R} |\nabla(U_2^{p/2})|^2 dx dt \\ + p(p-1) \int_0^t \int_{B_R} \psi(U_2)(-\Delta)^{1-s} U_2 dx dt \leq \int_{B_R} u_0^p(x) dx. \end{aligned}$$

4. **(Second energy estimate)** For all $0 < t < T$ we have

$$(4.11) \quad \begin{aligned} \frac{1}{2} \int_{B_R} |(-\Delta)^{-\frac{s}{2}} U_2(t)|^2 dx + \delta \int_0^t \int_{B_R} |(-\Delta)^{\frac{1-s}{2}} [U_2]|^2 dx dt \\ + \int_0^t \int_{B_R} (U_2 + \mu)^{m-1} |\nabla(-\Delta)^{-s} U_2(t)|^2 dx dt \leq \frac{1}{2} \int_{B_R} |(-\Delta)^{-\frac{s}{2}} u_0|^2 dx. \end{aligned}$$

4.3 Limit as $R \rightarrow \infty$

In this section we argue for weak solutions $U_2 = (U_2)_R$ of Problem $(P_{\delta\mu R})$. The energy estimates (4.10) and (4.11) will give us sufficient information to accomplish the limits.

4.3.1 Existence of a limit

We remark that the integrals in B_R can be interpreted like integrals on whole \mathbb{R}^N since we have chosen U_2 to be zero outside B_R . Moreover, we can get, from the energy estimates (4.10) and (4.11), upper bounds which are independent on R . Note that the compactness technique used (see Lemma 7.5) requires compact embeddings, which motivates us to work on bounded domains.

I. Local existence of a limit. Let $\rho > 0$ and consider the ball $B_\rho \subset \mathbb{R}^N$. From (4.10) with $p = 2$ we get that $U_2 \in L^2(0, T : H^1(B_\rho))$ uniformly in $R > 0$ and then $\delta\Delta U_2 \in L^2(0, T : H^{-1}(\mathbb{R}^N))$. Also, (4.11) gives $U_2 \in L^2(0, T : H^{1-s}(B_\rho))$. From (4.11) we get that $\nabla \cdot ((U_2 + \mu)^{m-1} \nabla (-\Delta)^{-s} U_2) \in L^2(0, T : H^{-1}(\mathbb{R}^N))$. Applying Lemma 7.5 in the context

$$H^{1-s}(B_\rho) \subset L^2(B_\rho) \subset H^{-1}(B_\rho),$$

and noting that the left hand side inclusion is compact, we obtain that there exists a limit function $V_\rho \in L^2(B_\rho \times (0, T))$ such that, up to sub-sequences,

$$(4.12) \quad U_2 \rightarrow V_\rho \quad \text{as } R \rightarrow \infty \quad \text{in } L^2(B_\rho \times (0, T)).$$

II. Finding a global limit. In order to define a global limit in $L^2(\mathbb{R}^N \times (0, T))$ we adapt the classical covering plus diagonal argument. Let $\bigcup_{k=1}^\infty B_{\rho_k}$, with $(\rho_k)_{k=1}^\infty \subset \mathbb{R}_{\geq 0}$, be a countable covering of \mathbb{R}^N . By (4.12) we obtain there exists a subsequence $(R_j)_{j=1}^\infty$ such that $U_2|_{B_{\rho_1}} \rightarrow V_{\rho_1}$ as $R_j \rightarrow \infty$ in $L^2(B_{\rho_1} \times (0, T))$ and $V_{\rho_1} : B_{\rho_1} \rightarrow \mathbb{R}$. Next, we perform a similar argument starting from the subsequence $(R_j)_{j=1}^\infty$ and $U_2|_{B_{\rho_2}}$ to get that there exists a sub-subsequence $(R_{j_k})_{k=1}^\infty \subset (R_j)_{j=1}^\infty$ such that $U_2|_{B_{\rho_2}} \rightarrow V_{\rho_2}$ as $R_{j_k} \rightarrow \infty$ in $L^2(B_{\rho_2} \times (0, T))$ and $V_{\rho_2} : B_{\rho_2} \rightarrow \mathbb{R}$. It is clear that $V_{\rho_1} = V_{\rho_2}$ in $B_{\rho_1} \cap B_{\rho_2}$. The argument continues for the remaining balls $B_{\rho_3}, B_{\rho_4}, \dots$. In the end we define the function $V : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $V|_{B_{\rho_k}} = V_{\rho_k}$ for $k \in \mathbb{N}_{>0}$. We denote this limit U_3 for better organization. Therefore, up to subsequences,

$$U_2 \rightarrow U_3 \quad \text{as } R \rightarrow \infty \quad \text{in } L^2(0, T : L^2_{\text{loc}}(\mathbb{R}^N)).$$

In particular, this implies $U_2 \rightarrow U_3$ as $R \rightarrow \infty$ a.e. in \mathbb{R}^N . We recall that the functions U_2 are extended by 0 in $\mathbb{R}^N \setminus B_R$ and then, by the energy estimate (4.10), we have that $\int_{\mathbb{R}^N} U_2^2 dx$ is uniformly bounded in $R > 0$. Then, by Fatou's Lemma we get that $U_3 \in L^2(\mathbb{R}^N \times (0, T))$ since

$$\liminf_{R \rightarrow \infty} \int_0^T \int_{\mathbb{R}^N} (U_2)^2 dx dt \geq \int_0^T \int_{\mathbb{R}^N} (U_3)^2 dx dt.$$

4.3.2 The limit U_3 is a solution of the new problem $(P_{\delta\mu})$

Similarly, one can prove that U_3 is a weak solution of Problem $(P_{\delta\mu})$:

$$(P_{\delta\mu}) \quad \begin{cases} (U_3)_t = \delta\Delta U_3 + \nabla \cdot ((U_3 + \mu)^{m-1} \nabla (-\Delta)^{-s} U_3) & \text{for } (x, t) \in \mathbb{R}^N \times (0, T), \\ U_3(x, 0) = \hat{u}_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

The test functions used in Subsection 4.2.1 are compactly supported so the arguments perfectly work here. Let ϕ be a suitable test function supported in a ball B_ρ for some $\rho > 0$. For the convergence of the nonlinear term we use that

$$(U_2 + \mu)^{m-1} \rightarrow (U_3 + \mu)^{m-1} \quad \text{in } L^2(B_\rho \times (0, T)) \quad \text{as } R \rightarrow +\infty,$$

and

$$(4.13) \quad \nabla(-\Delta)^{-s}U_2 \rightharpoonup \nabla(-\Delta)^{-s}U_3 \quad \text{in } L^2(B_\rho \times (0, T)) \text{ as } R \rightarrow +\infty,$$

where (4.13) is proved as in Lemma 4.3.

4.3.3 Energy estimates

All the energy estimates of U_2 can be written with integrals in \mathbb{R}^N and they provide upper bounds which independent on R . As before, the existence of a pointwise limit plus Fatou's Lemma allow us to pass to the limit as $R \rightarrow +\infty$. We refer to [32] for the proof of mass conservation. However, in Theorem 5.2 we prove this result in the general setting of measure data. We conclude with the following theorem.

Theorem 4.5. *Let $s \in (0, 1)$, $1 < m < \infty$ and $N \geq 1$. There exists a weak solution U_3 of Problem $(P_{\delta\mu})$ with initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, U_3 has the following properties:*

1. **(Conservation of total mass)** For all $0 < t < T$ we have $\int_{\mathbb{R}^N} U_3(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx$.
2. **(L^∞ estimate)** For all $0 < t < T$ we have $\|U_3(\cdot, t)\|_\infty \leq \|u_0\|_\infty$.
3. **(L^p energy estimate)** For all $1 < p < \infty$ and $0 < t < T$ we have

$$(4.14) \quad \begin{aligned} \int_{\mathbb{R}^N} U_3^p(x, t) dx + \frac{4(p-1)\delta}{p} \int_0^t \int_{\mathbb{R}^N} \left| \nabla(U_3^{p/2}) \right|^2 dx dt \\ + p(p-1) \int_0^t \int_{\mathbb{R}^N} \psi(U_3)(-\Delta)^{1-s} U_3 dx dt \leq \int_{\mathbb{R}^N} u_0^p(x) dx. \end{aligned}$$

4. **(Second energy estimate)** For all $0 < t < T$ we have

$$(4.15) \quad \begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} U_3(t) \right|^2 dx + \delta \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} [U_3] \right|^2 dx dt \\ + \int_0^t \int_{\mathbb{R}^N} (U_3 + \mu)^{m-1} \left| \nabla(-\Delta)^{-s} U_3(t) \right|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 dx. \end{aligned}$$

4.4 Limit as $\mu \rightarrow 0$

We remark that some of previous arguments can not be applied here since $(U_3 + \mu)^{-(m-1)}$ may degenerate as $\mu \rightarrow 0$ close to the free boundary. Therefore we adapt the proof to overcome this issue.

4.4.1 Existence of a limit

The energy estimates (4.14) and (4.15) gives us uniform upper bounds in μ which allows us to prove the existence of a limit

$$(4.16) \quad U_3 \rightarrow U_4 \quad \text{as } \mu \rightarrow 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N \times (0, T)),$$

using the same covering plus diagonal argument of Section 4.3.

4.4.2 The limit U_4 is a solution of the new problem (P_δ)

As before the compact support of the test functions allows us to prove that U_4 is in fact a weak solution of the problem:

$$(P_\delta) \quad \begin{cases} (U_4)_t = \delta \Delta U_4 + \nabla \cdot (U_4^{m-1} \nabla (-\Delta)^{-s} U_4) & \text{for } (x, t) \in \mathbb{R}^N \times (0, T), \\ U_4(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

The first integral of the weak formulation passes to the limit like in (4.6) as consequence of (4.16). It remains to prove that

$$(4.17) \quad \int_0^T \int_{\mathbb{R}^N} (U_3 + \mu)^{m-1} \nabla (-\Delta)^{-s} U_3 \cdot \nabla \phi dx dt \xrightarrow{\mu \rightarrow 0} \int_0^T \int_{\mathbb{R}^N} U_4^{m-1} \nabla (-\Delta)^{-s} U_4 \cdot \nabla \phi dx dt.$$

Let ϕ be supported in B_ρ for some $\rho > 0$. It is clear that

$$(4.18) \quad (U_3 + \mu)^{m-1} \rightarrow U_4^{m-1} \quad \text{as } \mu \rightarrow 0 \quad \text{in } L^2(B_\rho \times (0, T)).$$

Moreover, from the second energy estimate, we get that there exists a weak limit of U_3 in $L^2(0, T : H^{1-s}(B_\rho))$. Furthermore, the limit can be identified in $L^2(B_\rho \times (0, T))$ from (4.16), and then

$$U_3 \rightharpoonup U_4 \quad \text{as } \mu \rightarrow 0 \quad \text{in } L^2(0, T : H^{1-s}(B_\rho)).$$

Since the term $\nabla(-\Delta)^{-s}$ is of order $1 - 2s$, which is smaller than $1 - s$, then

$$(4.19) \quad \nabla(-\Delta)^{-s} U_3 \rightharpoonup \nabla(-\Delta)^{-s} U_4 \quad \text{in } L^2(B_\rho \times (0, T)).$$

Combining (4.18) and (4.19) the convergence (4.17) follows.

4.4.3 Energy estimates

We state the main properties of the solution of Problem (P_δ) .

Theorem 4.6. *Let $s \in (0, 1)$, $1 < m < \infty$ and $N \geq 1$. There exists a weak solution U_4 of Problem (P_δ) with initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, U_4 has the following properties:*

1. **(Conservation of total mass)** For all $0 < t < T$ we have $\int_{\mathbb{R}^N} U_4(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx$.
2. **(L^∞ -estimate)** For all $0 < t < T$ we have $\|U_4(\cdot, t)\|_\infty \leq \|u_0\|_\infty$.
3. **(L^p -decay energy estimate)** For all $1 < p < \infty$ and $0 < t < T$

$$(4.20) \quad \begin{aligned} \int_{\mathbb{R}^N} U_4^p(x, t) dx + \frac{4(p-1)\delta}{p} \int_0^t \int_{\mathbb{R}^N} \left| \nabla(U_4^{p/2}) \right|^2 dx dt \\ + \frac{p(p-1)}{m+p-2} \int_0^t \int_{\mathbb{R}^N} U_4^{m+p-2} (-\Delta)^{1-s} U_4 dx dt \leq \int_{\mathbb{R}^N} u_0^p(x) dx. \end{aligned}$$

4. (**Second energy estimate**) For all $0 < t < T$ we have

$$(4.21) \quad \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} U_4(t) \right|^2 dx + \delta \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} [U_4] \right|^2 dx dt \\ + \int_0^t \int_{\mathbb{R}^N} U_4^{m-1} |\nabla (-\Delta)^{-s} U_4(t)|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 dx.$$

The proof is as in the previous part. The term $\int_0^t \int_{\mathbb{R}^N} (U_3 + \mu)^{m-1} |\nabla (-\Delta)^{-s} U_4(t)|^2 dx dt$ passes to the limit by Fatou's Lemma since $(U_3 + \mu)^{m-1} \rightarrow U_4^{m-1}$ as $\mu \rightarrow 0$ pointwise.

4.5 Limit as $\delta \rightarrow 0$

This part is quite interesting and brings some novelty in the techniques we have employed so far. Here we use a different compactness criteria in order to derive the convergence as $\delta \rightarrow 0$. This is a consequence of the lack of regularity that was given by the δ -term in the previous approximating problems.

Estimates (4.20) and (4.21) provide an upper bound independent of δ . The terms with δ coefficient are positive and bounded and therefore U_4 satisfies:

$$(4.22) \quad \int_{\mathbb{R}^N} U_4^p(x, t) dx + \frac{p(p-1)}{m+p-2} \int_0^t \int_{\mathbb{R}^N} U_4^{m+p-2} (-\Delta)^{1-s} U_4 dx dt \leq \int_{\mathbb{R}^N} u_0^p(x) dx,$$

and

$$(4.23) \quad \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} U_4(t) \right|^2 dx + \int_0^t \int_{\mathbb{R}^N} U_4^{m-1} |\nabla (-\Delta)^{-s} U_4(t)|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 dx.$$

4.5.1 Existence of a limit. Compactness estimate II

We will prove compactness for the following sequence:

$$W_\delta := \begin{cases} U_4 & \text{if } m \leq 2 \\ U_4^m & \text{if } m > 2. \end{cases}$$

The idea is to apply Theorem 7.8 for W_δ and in order to use this compactness criteria we need to work on a bounded domain B_ρ for $\rho > 0$. From (4.22), applying Stroock-Varopoulos we obtain

$$(4.24) \quad \int_{\mathbb{R}^N} u_0^p(x) dx \geq \int_{\mathbb{R}^N} U_4^p(x, t) dx + \frac{4p(p-1)}{(m+p-1)^2} \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} U_4^{\frac{m+p-1}{2}} \right|^2 dx dt.$$

In this way we get a uniform bound for W_δ in $L^2(0, T : H^{1-s}(B_\rho))$ by using (4.24) with $p = 3 - m$ if $m \leq 2$ and $p = m + 1$ if $m > 2$. Note that the exponent $3 - m$ is again critical in the proof of existence, as happened in the article [32]. In both cases we get that there exists a weak limit

$$W_\delta \rightharpoonup W \text{ in } L^2(0, T : H^{1-s}(B_\rho)).$$

Then, hypothesis a) in Theorem 7.8 is satisfied in the context $V = H^{1-s}(B_\rho)$ and $H = L^2(B_\rho)$. However, b) also holds due to the energy estimate (4.24) for $p = 2q$ where $q = 1$ if $m \leq 2$ and $q = m$ if $m > 2$. Indeed we have the following estimate

$$\sup_{\delta > 0} \|W_\delta(t)\|_{L^2(B_\rho)} = \sup_{\delta > 0} \|U_4^q(t)\|_{L^2(B_\rho)} = \sup_{\delta > 0} \|U_4(t)\|_{L^{2q}(B_\rho)}^q \leq \|u_0\|_{L^{2q}(B_\rho)}^q < +\infty$$

for every $t \in (0, T)$. It remains to prove assumption c) of Theorem 7.8. Since $L^2(B_\rho)$ is a separable Hilbert space, we can find a countable set D dense in $L^2(B_\rho)$. Moreover, we can assume that the elements $\psi \in D$ are smooth and nonnegative.

We want to prove that the family of functions $g_\psi^\delta(t) := \langle U_4(\cdot, t), \psi \rangle_{L^2(B_\rho)}$ is relatively compact in $L^1((0, T))$. First, $\{g_\psi^\delta\}_{\delta > 0}$ is equibounded in $L^1((0, T))$ since

$$\begin{aligned} \|g_\psi^\delta\|_{L^1((0, T))} &:= \int_0^T \int_{B_\rho} U_4(x, t) \psi(x) dx dt \leq \left(\int_0^T \int_{B_\rho} (U_4)^2 dx dt \right)^{1/2} \left(\int_0^T \int_{B_\rho} \psi^2 dx dt \right)^{1/2} \\ &\leq T \|u_0\|_{L^2(B_\rho)} \|\psi\|_{L^2(B_\rho)}. \end{aligned}$$

Moreover, we also have that $g_\psi^\delta(t)$ is equicontinuous in $L^1((0, T))$: using (P_δ) we have

$$\begin{aligned} \int_0^T (g_\psi^\delta)'(t) dt &= \delta \int_0^T \langle U_4, \Delta \psi \rangle dt + \int_0^T \langle (-\Delta)^{-\frac{s}{2}} U_4, \nabla (-\Delta)^{-\frac{s}{2}} (U_4^{m-1} \nabla \psi) \rangle dt \\ &\leq \delta \|U_4\|_{L^2(B_\rho \times (0, T))} T^{1/2} \|\Delta \psi\|_{L^2(B_\rho)} + \|(-\Delta)^{-\frac{s}{2}} U_4\|_{L^2(B_\rho \times (0, T))} \|\nabla (-\Delta)^{-\frac{s}{2}} (U_4^{m-1} \nabla \psi)\|_{L^2(B_\rho \times (0, T))}, \end{aligned}$$

where all the terms in the last inequality are absolutely bounded in δ due to the energy estimates (4.22) and (4.23). We use the fact that for any smooth function $\psi \in D$ we have that $\psi U_4^{m-1} \in L^2(0, T : H^{1-s}(\mathbb{R}^N))$ and then $\nabla (-\Delta)^{-\frac{s}{2}} (U_4^{m-1} \nabla \psi) \in L^2(\mathbb{R}^N \times (0, T))$ uniformly on δ .

In this way, if $m \leq 2$, since $U_4 = W_\delta$, we have that hypothesis c) of Theorem 7.8 is satisfied by W_δ . If $m \geq 2$, then $\langle U_4^m, \psi \rangle_{L^2(B_\rho)}$ is clearly equibounded in $L^1((0, T))$. Moreover, by the equicontinuity of $g_\psi^\delta(t)$ and the following estimate

$$\int_{t_1}^{t_2} \langle U_4^m, \psi \rangle_{L^2(B_\rho)} dt \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}^{m-1} \int_{t_1}^{t_2} \langle U_4, \psi \rangle_{L^2(\Omega)} dt,$$

we have that $\langle U_4^m, \psi \rangle_{L^2(B_\rho)}$ is also equicontinuous in $L^1((0, T))$. We apply Theorem 7.6 to obtain

$$W_\delta \rightarrow W \text{ in } L^2(B_\rho \times (0, T)).$$

For $m \leq 2$ this means $U_4 \rightarrow W$ in $L^2(B_\rho \times (0, T))$ and we are done. Now, let $m > 2$. We have $W_\delta = U_4^m \rightarrow W$ in $L^2(B_\rho \times (0, T))$. Since $(U_4)_\delta \in L^\infty(\mathbb{R}^N \times (0, T))$ uniformly in δ then also the limit $W(x, t) \in L^\infty(\mathbb{R}^N \times (0, T))$. In both cases, by the covering plus diagonal argument and Fatou's Lemma as in Section 4.3.1, we obtain, up to a subsequence, that

$$(4.25) \quad U_4 \rightarrow u \text{ in } L_{\text{loc}}^2(\mathbb{R}^N \times (0, T)).$$

4.5.2 The limit u is a weak solution of Problem (1.1)

We pass to the limit as $\delta \rightarrow 0$ in the weak formulation corresponding to Problem (P_δ) . Let ϕ a compactly supported test function with support in B_ρ . Then by (4.25) we get

$$\int_0^T \int_{\mathbb{R}^N} U_4 \phi_t dx dt \rightarrow \int_0^T \int_{\mathbb{R}^N} u \phi_t dx dt \quad \text{as } \delta \rightarrow 0.$$

Moreover,

$$\delta \int_0^T \int_{\mathbb{R}^N} U_4 \Delta \phi dx dt \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

It remains to prove that

$$(4.26) \quad \int_0^T \int_{\mathbb{R}^N} U_4^{m-1} \nabla(-\Delta)^{-s} U_4 \nabla \phi dx dt \rightarrow \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla(-\Delta)^{-s} u \nabla \phi dx dt.$$

I. Case $m \leq 2$. From L^p estimate (4.24) with $p = 3 - m$ we have that $U_4 \in H^{1-s}(B_\rho)$ and then $U_4 \rightharpoonup u$ in $H^{1-s}(\Omega)$. As a consequence

$$(4.27) \quad \nabla(-\Delta)^{-s} U_4 \rightharpoonup \nabla(-\Delta)^{-s} u \quad \text{in } L^2(B_\rho \times (0, T)).$$

Moreover, we have that $U_4^{m-1} \rightarrow u^{m-1}$ in $L^2(B_\rho \times (0, T))$, which together with (4.27) implies (4.26).

II. Case $m > 2$. We use Lemma 6.5 and Lemma 6.9 from [2] for $v := U_4(\nabla \phi)^{\frac{1}{m-1}}$ to get that $\nabla \cdot (-\Delta)^{-s}(U_4^{m-1} \nabla \phi) \in L^p(\mathbb{R}^N \times (0, T))$ uniformly on δ , for a certain $p > 1$. On the other hand, $U_4 \in L^q(\mathbb{R}^N)$ for any $L^q(\mathbb{R}^N)$ uniformly on $\delta > 0$ and thus we integrate by parts the first integral of (4.26) to get

$$I(U_4) := \int_0^T \int_{\mathbb{R}^N} U_4^{m-1} \nabla(-\Delta)^{-s} U_4 \nabla \phi dx dt = \int_0^T \int_{\mathbb{R}^N} U_4 \nabla \cdot (-\Delta)^{-s}(U_4^{m-1} \nabla \phi) dx dt.$$

Moreover, for every ϕ there exists a weak limit

$$(4.28) \quad \nabla \cdot (-\Delta)^{-s}(U_4^{m-1} \nabla \phi) \rightharpoonup v \quad \text{as } \delta \rightarrow 0 \quad \text{in } L^p(\mathbb{R}^N \times (0, T)).$$

We identify the limit in the sense of distributions and show that $v = \nabla \cdot (-\Delta)^{-s}(u^{m-1} \nabla \phi)$: indeed we have that

$$\int_0^T \int_{\mathbb{R}^N} U_4^{m-1} \nabla \phi \nabla(-\Delta)^{-s} \psi dx dt \rightarrow \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla \phi \nabla(-\Delta)^{-s} \psi dx dt \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N \times (0, T))$$

since $U_4^{m-1} \rightarrow u^{m-1}$ in $L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$. Therefore

$$(4.29) \quad \nabla \cdot (-\Delta)^{-s}(U_4^{m-1} \nabla \phi) \rightharpoonup \nabla \cdot (-\Delta)^{-s}(u^{m-1} \nabla \phi) \quad \text{as } \delta \rightarrow 0 \quad \text{in } L^p(\mathbb{R}^N \times (0, T)),$$

for every test function ϕ .

Let $R > 0$. Then

$$\begin{aligned} I(U_4) &= \int_0^T \int_{B_R} U_4 \nabla \cdot (-\Delta)^{-s}(U_4^{m-1} \nabla \phi) dx dt + \int_0^T \int_{\mathbb{R}^N \setminus B_R} U_4 \nabla \cdot (-\Delta)^{-s}(U_4^{m-1} \nabla \phi) dx dt \\ &= I_1(U_4) + I_2(U_4). \end{aligned}$$

Since the sequence $U_4^{m-1}\nabla\phi$ has the same compact support for all δ then $\nabla \cdot (-\Delta)^{-s}(U_4^{m-1}\nabla\phi)$ uniformly decays for large $|x|$ (see (4.30)). Then we can choose R big enough such that $I_2(U_4) < \epsilon/3$. In the same way $I_2(u) < \epsilon/3$. Now, with this given R we use that $U_4 \rightarrow u$ in $L_{\text{loc}}^q(\mathbb{R}^N \times (0, T))$ together with (4.29) and we have $I_1(U_4) \rightarrow I_1(u)$ as $\delta \rightarrow 0$. Thus, we choose $\delta > 0$ such that

$$|I(U_4) - I(u)| \leq |I_1(U_4) - I_1(u)| + |I_2(U_4)| + |I_2(u)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

We integrate by parts to obtain the desired convergence (4.26).

4.5.3 Energy estimates

We pass to the limit in the energy estimates. From (4.22)-(4.24) we get that

$$\int_{\mathbb{R}^N} u^p(x, t) dx + \frac{p(p-1)}{m+p-2} \int_0^t \int_{\mathbb{R}^N} u^{m+p-2} (-\Delta)^{1-s} u \, dx dt \leq \int_{\mathbb{R}^N} u_0^p(x) dx.$$

From (4.23) we get

$$\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(t) \right|^2 dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} |\nabla (-\Delta)^{-s} u(t)|^2 \, dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 dx.$$

We have obtained so far the existence of a weak solution of Problem (1.1) enjoying regularity properties and the corresponding energy estimates. This concludes the proof of Theorem 2.2.

4.6 Dealing with the case $N = 1$, $s \in [\frac{1}{2}, 1)$

The operator $(-\Delta)^{-s}$ is not well defined when $N = 1$ and $\frac{1}{2} < s < 1$ since the convolution kernel $K_s = \frac{1}{|x|^{1-2s}}$ does not decay at infinity. Therefore it does not make sense to think of equation (1.1) in terms of a pressure. This may not be very convenient, but the issue can be avoided by writing the equation as

$$u_t = \nabla \cdot (u^{m-1} \nabla^{1-2s} u),$$

where ∇^{1-2s} denotes formally the composition operator $\nabla(-\Delta)^{-s}$. According to [2], ∇^{1-2s} can be written in the whole range $0 < s < 1$ in terms of the singular integral formula for smooth and bounded functions

$$(4.30) \quad \nabla^{1-2s} \psi(x) = C_{N,s} \int (\psi(x) - \psi(x+z)) \frac{\text{sign}(z)}{|z|^{N+1-2s}} dz.$$

Note that for $\frac{1}{2} < s < 1$, $|z|^{-N-1+2s} \in L_{\text{loc}}^1(\mathbb{R}^N)$ and decays at infinity. Note also that ∇^{1-2s} has the Fourier symbol given by $i \text{sign}(\xi) |\xi|^{1-2s}$. Moreover, the operator $(-\Delta)^{-\frac{s}{2}}$ is well defined in the whole range $0 < s < 1$ even in dimension $N = 1$. In this way, we have the following property:

$$\nabla^{1-2s} = (-\Delta)^{-\frac{s}{2}} \nabla (-\Delta)^{-\frac{s}{2}} = (-\Delta)^{-\frac{s}{2}} \nabla^{1-s}.$$

The L^p energy estimate (2.1) still has the same form, while the second energy estimate (2.2) needs has to be reformulated as

$$\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(t) \right|^2 dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} |\nabla^{1-2s} u(t)|^2 \, dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_0 \right|^2 dx.$$

The proofs of Section 4 follow similarly. For the $\epsilon \rightarrow 0$ limit, we shall use part b) of Lemma 3.3.

5 Existence of solutions with measure data

In this section we give the proof of the existence of weak solutions taking as initial data any $\mu \in \mathcal{M}^+(\mathbb{R}^N)$, the space of nonnegative Radon measures on \mathbb{R}^N with *finite mass*. In particular, this includes the case of only integrable data $u_0 \in L^1(\mathbb{R}^N)$. Therefore, we improve the results from [9, 32] to less restrictive initial data. As precedent we mention [7] where the authors extend the existence theory for $m = 2$ to every $u_0 \in L^1(\mathbb{R}^N)$. The case of measures has been considered for the case $m = 2$, $s \rightarrow 1$ in [28], and for model (1.2) in [36].

Definition 5.1. Let $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. We say that $u \geq 0$ is a weak solution of Problem (1.1) with initial data μ if:

(i) $u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$, (ii) $\nabla(-\Delta)^{-s}u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$, (iii) $u^{m-1}\nabla(-\Delta)^{-s}u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} u \phi_t dx dt - \int_0^T \int_{\mathbb{R}^N} u^{m-1} \nabla(-\Delta)^{-s}u \cdot \nabla \phi dx dt + \int_{\mathbb{R}^N} \phi(x, 0) d\mu(x) = 0.$$

for all test functions $\phi \in C_c^1(\mathbb{R}^N \times [0, T))$.

Theorem 5.2. Let $1 < m < \infty$, $N \geq 1$ and $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Then there exists a weak solution $u \geq 0$ (in the sense of Definition 5.1) of Problem (1.1) such that the smoothing effect (2.3) holds for $p = 1$ in the following sense:

$$(5.1) \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m} t^{-\gamma} \mu(\mathbb{R}^N)^\delta \quad \text{for all } t > 0,$$

where $\gamma = \frac{N}{(m-1)N+2(1-s)}$, $\delta = \frac{2(1-s)}{(m-1)N+2(1-s)}$. Moreover,

$$u \in L^\infty((\tau, \infty) : L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (\tau, \infty)) \cap L^\infty((0, \infty) : \mathcal{M}^+(\mathbb{R}^N)) \quad \text{for all } \tau > 0$$

and it has the following properties

1. **(Conservation of mass)** For all $0 < t < T$ we have $\int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} d\mu(x)$.

2. **(L^p energy estimate)** For all $1 < p < \infty$ and $0 < \tau < t < T$ we have

$$\int_{\mathbb{R}^N} u^p(x, t) dx + \frac{4p(p-1)}{(m+p-1)^2} \int_\tau^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-1}{2}} \right|^2 dx dt \leq \int_{\mathbb{R}^N} u^p(x, \tau) dx.$$

3. **(Second energy estimate)** For all $0 < \tau < t < T$ we have

$$\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(t) \right|^2 dx + \int_\tau^t \int_{\mathbb{R}^N} u^{m-1} \left| \nabla(-\Delta)^{-s} u(t) \right|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u(\tau) \right|^2 dx.$$

Remark 5. If μ is an absolutely continuous with respect to the Lebesgue measure, it has a density $u_0 \in L^1(\mathbb{R}^N)$ such that $d\mu(x) = u_0(x)dx$. In this case u_0 is an initial condition in the sense given in Definition 2.1.

Proof. I. Approximation with bounded solutions. Let $\{\rho_n\}_{n>0}$ be a sequence of standard mollifiers. We define the approximate initial data by convolution, i.e., for any $n > 0$ we consider the function $(u_0)_n \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ defined by

$$(u_0)_n(x) := \int_{\mathbb{R}^N} \rho_n(x-z) d\mu(z).$$

Note that, by Fubini's Theorem, we have that

$$(5.2) \quad \|(u_0)_n\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} d\mu(z) = \mu(\mathbb{R}^N).$$

It is clear that $(u_0)_n \rightarrow \mu$ as $n \rightarrow \infty$ in the sense required by Definition 5.1, that is,

$$(5.3) \quad \int_{\mathbb{R}^N} (u_0)_n(x) \psi(x) dx \rightarrow \int_{\mathbb{R}^N} \psi(x) d\mu(x) \quad \text{as } n \rightarrow \infty,$$

for all $\psi \in C_c^1(\mathbb{R}^N)$. Now let $u_n \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be the solution of Problem (1.1) with initial data $(u_0)_n$ provided by Theorem 2.2. Moreover, thanks to the L^1 - L^∞ smoothing effect given by Theorem 2.3 we have the following estimates that are independent of n :

- i) For all $0 < t < T$ we have $\|u_n(\cdot, t)\|_{L^1(\mathbb{R}^N)} = \|(u_0)_n\|_{L^1(\mathbb{R}^N)} = \mu(\mathbb{R}^N)$.
- ii) For all $0 < \tau < t \leq T$ we have

$$\|u_n(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_n(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m} \tau^{-\gamma} \|(u_0)_n\|_{L^1(\mathbb{R}^N)}^\delta = C_{N,s,m} \tau^{-\gamma} \mu(\mathbb{R}^N)^\delta,$$

where $\gamma = \frac{N}{(m-1)N+2(1-s)}$, $\delta = \frac{2(1-s)}{(m-1)N+2(1-s)}$.

Furthermore, since i) and ii) show that $u_n \in L^\infty(\mathbb{R}^N \times (\tau, T)) \cap L^1(\mathbb{R}^N \times (0, T))$ uniformly in n , we have the following energy estimates for which the right hand side are absolutely bounded in n (the precise bounds will be given later):

- iii) For all $1 < p < \infty$ and $0 < \tau < t \leq T$,

$$\int_{\mathbb{R}^N} u_n^p(x, t) dx + \frac{4p(p-1)}{(m+p-1)^2} \int_\tau^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} u_n^{\frac{m+p-1}{2}} \right|^2 dx dt \leq \int_{\mathbb{R}^N} u_n^p(x, \tau) dx.$$

- iv) For all $0 < \tau < t \leq T$,

$$\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_n(t) \right|^2 dx + \int_\tau^t \int_{\mathbb{R}^N} u_n^{m-1} |\nabla (-\Delta)^{-s} u_n(t)|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_n(\tau) \right|^2 dx.$$

II. Convergence away from $t = 0$. Given any $\tau > 0$ we can use the compactness criteria given by Theorem 7.8 as in Section 4.5.1 to show that

$$(5.4) \quad u_n \rightharpoonup u^\tau \quad \text{as } n \rightarrow \infty \quad \text{in } L_{\text{loc}}^2(\mathbb{R}^N \times (\tau, T)).$$

In the weak formulation, for any $\phi \in C_c^\infty(\mathbb{R}^N \times [0, T])$, u_n satisfies:

$$\int_\tau^T \int_{\mathbb{R}^N} u_n \phi_t dx dt - \int_\tau^T \int_{\mathbb{R}^N} u_n^{m-1} \nabla (-\Delta)^{-s} u_n \nabla \phi dx dt + \int_{\mathbb{R}^N} u_n(\tau) \phi(x, \tau) dx = 0.$$

Moreover, we can proceed as in Section 4.5.2 to prove that for any test function ϕ we have

$$\int_{\tau}^T \int_{\mathbb{R}^N} u_n \phi_t dx dt \rightarrow \int_{\tau}^T \int_{\mathbb{R}^N} u^{\tau} \phi_t dx dt \quad \text{as } \delta \rightarrow 0.$$

and

$$\int_{\tau}^T \int_{\mathbb{R}^N} u_n^{m-1} \nabla(-\Delta)^{-s} u_n \nabla \phi dx dt \rightarrow \int_{\tau}^T \int_{\mathbb{R}^N} (u^{\tau})^{m-1} \nabla(-\Delta)^{-s} u^{\tau} \nabla \phi dx dt.$$

III. Uniform estimates at $t = 0$. In order to show that we can pass to the limit as $\tau \rightarrow 0$ to obtain a weak solution of Problem (1.1) we need to prove that the remaining terms converge to zero as $\tau \rightarrow 0$. First of all,

$$\left| \int_0^{\tau} \int_{\mathbb{R}^N} u_n \phi_t dx dt \right| \leq C \int_0^{\tau} \|u_n(\cdot, t)\|_{L^1(\mathbb{R}^N)} dt = C\tau \mu(\mathbb{R}^N).$$

Now we use the classical Riesz embedding (c.f [33]) and that $u_n(\cdot, t) \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ for any $t > 0$ to get

$$\int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_n(t) \right|^2 dx \leq C \|u_n(t)\|_p^2 \quad \text{with} \quad \frac{1}{2} = \frac{1}{p} - \frac{s}{N}.$$

Also, from the smoothing effect, we have

$$\|u_n(t)\|_p^p \leq \|u_n(t)\|_1 \|u_n(t)\|_{\infty}^{p-1} \leq C \mu(\mathbb{R}^N)^{1+(p-1)\delta} t^{-\gamma(p-1)}.$$

In this way, we get

$$\int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_n(t) \right|^2 dx \leq C \mu(\mathbb{R}^N)^{\sigma} t^{-\lambda},$$

for some $\sigma > 0$ and

$$\lambda = \frac{2\gamma(p-1)}{p} = \frac{2N}{(m-1)N+2-2s} \frac{N-2s}{2N} = \frac{N-2s}{(m-1)N+2-2s}.$$

Consider the strip $Q_k = \mathbb{R}^N \times (t_k, t_{k-1})$ with $t_k = 2^{-k}$. Then

$$\begin{aligned} \int \int_{Q_k} u_n^{m-1} |\nabla(-\Delta)^{-s} u_n| dx dt &\leq \left(\int \int_{Q_k} u_n^{m-1} dx dt \right)^{1/2} \left(\int \int_{Q_k} u_n^{m-1} |\nabla(-\Delta)^{-s} u_n|^2 dx dt \right)^{1/2} \\ &\leq \|u_n(t)\|_{L^{\frac{m-2}{2}}(Q_k)}^{\frac{m-2}{2}} (\mu(\mathbb{R}^N) t_k)^{1/2} \left(\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{-\frac{s}{2}} u_n(t_k) \right|^2 dx \right)^{1/2} \\ &\leq C \mu(\mathbb{R}^N)^{\hat{\sigma}} t_k^{-\gamma \frac{m-2}{2}} t_k^{\frac{1}{2}} t_k^{-\frac{\lambda}{2}} \\ &= C \mu(\mathbb{R}^N)^{\hat{\sigma}} t_k^{\alpha}, \end{aligned}$$

for some $\hat{\sigma} > 0$ and

$$\alpha = \frac{1}{2} \left(1 - \gamma(m-2) - \frac{N-2s}{(m-1)N+2-2s} \right) = \frac{1}{(m-1)N+2-2s} > 0.$$

In this way,

$$(5.5) \quad \left| \int_0^{\tau} \int_{\mathbb{R}^N} u_n^{m-1} \nabla(-\Delta)^{-s} u_n \nabla \phi dx dt \right| \leq \|\nabla \phi\|_{\infty} \int_0^{\tau} \int_{\mathbb{R}^N} u_n^{m-1} |\nabla(-\Delta)^{-s} u_n| dx dt \leq \Lambda(\tau)$$

for some modulus of continuity Λ .

IV. Initial data. The only thing left is to prove that the initial data is taken. Let ϕ be a $C_c^1(\mathbb{R}^N)$ test function. Then, using the estimate given by (5.5), we get

$$(5.6) \quad \left| \int_{\mathbb{R}^N} (u_n(\tau) - (u_0)_n) \phi dx \right| = \left| \int_0^\tau \int_{\mathbb{R}^N} \partial_t u_n \phi dx dt \right| = \left| \int_0^\tau \int_{\mathbb{R}^N} u_n^{m-1} \nabla(-\Delta)^{-s} u_n \nabla \phi dx dt \right| \leq \Lambda(\tau).$$

A standard diagonal procedure in n and τ concludes the proof.

V. Conservation of mass. We can also conclude conservation of mass by taking a sequence of test functions of the cutoff type, $\phi_R(x) = \phi(x/R)$ with $0 \leq \phi \leq 1$ and $\phi_1(x) = 1$ for $|x| \leq 1$ and such that $\|\nabla \phi_R\|_{L^\infty(\mathbb{R}^N)} = O(R^{-1})$ (see appendix A.2 in [32] for more details). Then, using (5.5) and (5.6), we get that for any $\tau > 0$ we have

$$\left| \int_{\mathbb{R}^N} u_n(\tau) \phi_R dx - \int_{\mathbb{R}^N} (u_0)_n \phi_R dx \right| \leq C \frac{\Lambda(\tau)}{R}.$$

In particular, the previous estimate implies that

$$\begin{aligned} \int_{\mathbb{R}^N} u_n(\tau) \phi_R dx &\geq \int_{\mathbb{R}^N} (u_0)_n \phi_R dx - C \Lambda(\tau)/R \\ &= \int_{\mathbb{R}^N} (u_0)_n \phi_R dx - \int_{\mathbb{R}^N} \phi_R(x) d\mu(x) + \int_{\mathbb{R}^N} \phi_R(x) d\mu(x) - C \Lambda(\tau)/R \end{aligned}$$

In view of (5.4) and (5.3) we can let $n \rightarrow \infty$ in the previous estimate to get

$$\int_{\mathbb{R}^N} u(\tau) \phi_R dx \geq \int_{\mathbb{R}^N} \phi_R(x) d\mu(x) - C \Lambda(\tau)/R.$$

Note that, since μ is measure with finite mass in \mathbb{R}^N , then

$$\int_{\mathbb{R}^N} \phi_R(x) d\mu(x) \geq \mu(\mathbb{R}^N) - \epsilon(R)$$

with $\epsilon(R) \rightarrow 0$ as $R \rightarrow \infty$. Therefore,

$$\int_{\mathbb{R}^N} u(\tau) \phi_R dx \geq \mu(\mathbb{R}^N) - \epsilon(R) - C \Lambda(\tau)/R.$$

Letting now $R \rightarrow \infty$ we get

$$\int_{\mathbb{R}^N} u(\tau) dx \geq \mu(\mathbb{R}^N).$$

In this way we show that no mass is lost at infinity during the evolution. The other inequality comes from the construction of solutions.

□

Remark 6. The proof of mass conservation given in Theorem 5.2 is strongly based on the estimates available from the $L^1 - L^\infty$ smoothing effect. This is a more powerful tool than the one presented in [32] where the assumption of the boundedness on solution was unavoidable.

6 Comments and open problems

• **First energy estimate.** Let u be the solution of Problem (1.1). The following formal estimates can be derived for any $t > 0$:

$$(6.1) \quad \begin{aligned} & |(2-m)(3-m)| \int_0^t \int_{\mathbb{R}^N} |\nabla(-\Delta)^{-\frac{s}{2}} u|^2 dx dt + \int_{\mathbb{R}^N} u(t)^{3-m} dx \leq \int_{\mathbb{R}^N} u_0^{3-m} dx \quad \text{if } m \neq 2, 3. \\ & \int_0^t \int_{\mathbb{R}^N} |\nabla(-\Delta)^{-\frac{s}{2}} u|^2 dx dt + \int_{\mathbb{R}^N} (u(t) - \log(u(t))) dx \leq \int_{\mathbb{R}^N} (u_0 - \log(u_0)) dx \quad \text{if } m = 3. \\ & \int_0^t \int_{\mathbb{R}^N} |\nabla(-\Delta)^{-\frac{s}{2}} u|^2 dx dt + \int_{\mathbb{R}^N} u(t) \log(u(t)) dx \leq \int_{\mathbb{R}^N} u_0 \log(u_0) dx \quad \text{if } m = 2. \end{aligned}$$

This kind of energy estimates were a key tool to prove existence in the previous paper [32]. When $m \in (1, 2)$, they only require $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ in order to have uniform bounds on the $L^2(\mathbb{R}^N \times (0, T))$ norm of $\nabla(-\Delta)^{-\frac{s}{2}} u$. When $m \in [2, 3]$ they are still being useful energy estimates, but an additional decay has to be imposed to u_0 . In [32] we proved that if u_0 decays exponentially for large $|x|$, then $u(t)$ has a similar decay and (6.1) gives us meaningful information. For $m \geq 3$, (6.1) is not valid anymore with a decay property. This has motivated us to use a different approximation technique in the present paper which satisfies a different energy estimate (2.1) without any additional conditions to be imposed on the initial data.

• The L^p -energy estimate (2.1) can be proved for a general nonlinearity $\varphi(u)$:

$$\int_{\mathbb{R}^N} \varphi(u)(x, t) dx + \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} \psi(u) \right|^2 dx dt \leq \int_{\mathbb{R}^N} \varphi(u_0)(x) dx,$$

where $(\psi')^2(a) = \varphi''(a)a^{m-1}$. This kind of energy estimate is used in [2] and in [15].

• **More general equations and estimates.** The techniques employed in this paper can be used to prove existence results for more general equations of the form

$$(6.2) \quad u_t(x, t) = \nabla \cdot (G'(u) \nabla(-\Delta)^{-s} u),$$

where $G : [0, +\infty) \rightarrow [0, +\infty)$ has at most linear growth at the origin or $G' > 0$. The general Stroock-Varopoulos Inequality (7.1) allows us to obtain an energy inequality also in this case:

$$\int_{\mathbb{R}^N} \varphi(u)(x, t) dx + \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} \psi(u) \right|^2 dx dt \leq \int_{\mathbb{R}^N} \varphi(u_0)(x) dx,$$

where $(\psi')^2(a) = \varphi''(a)G'(a)$. We give a few examples below.

a) For instance we consider $G(u) = \frac{1}{m}(u+1)^m$, then $G'(u) = (u+1)^{m-1}$ and the model is

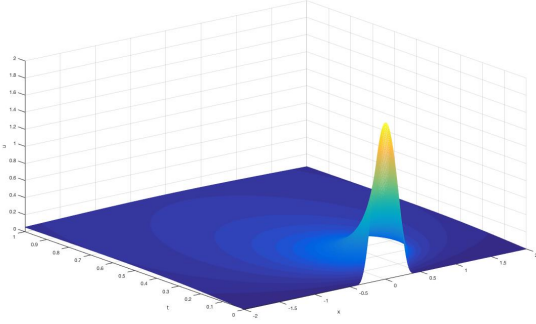
$$(6.3) \quad u_t(x, t) = \nabla \cdot ((u+1)^{m-1} \nabla(-\Delta)^{-s} u).$$

This corresponds to the approximating problem ($P_{\delta\mu}$) without viscosity $\mu = 1$, $\delta = 0$. There is positive velocity and the solutions seem to have infinite speed of propagation. See Figure 1a for the particular case $m = 2$.

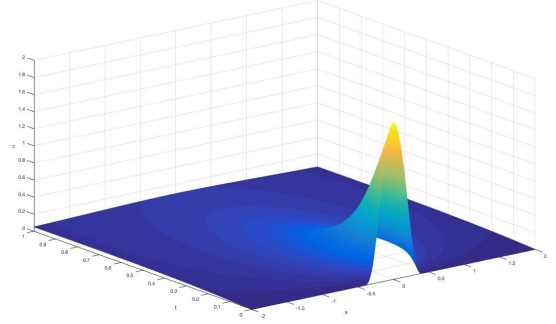
b) Let $G(u) = \log(1+u)$, then $G'(u) = \frac{1}{1+u}$ and the model is

$$(6.4) \quad u_t(x, t) = \nabla \cdot \left(\frac{1}{1+u} \nabla(-\Delta)^{-s} u \right).$$

We provide a numerical simulation in Figure 1b. This may correspond to $m \rightarrow 0$, $m > 0$. This nonlinearity has been considered for the Fractional Porous Medium Equation $u_t + (-\Delta)^s \log(1+u) = 0$ in [25].



(a) Solution for model (6.3) with $s = 0.5$



(b) Solution for model (6.4) with $s = 0.5$

Figure 1: More general equations of type (6.2)

• **Finite/infinite speed of propagation depending on the nonlinearity.** In [32] some preliminary results have been obtained concerning the positivity properties of the solution of Problem (1.1). Jointly with the existence theory developed in the present work for all $1 < m < \infty$ we have the following results so far:

a) Let $N \geq 1$, $m \in [2, +\infty)$, $s \in (0, 1)$ and let u be a constructed weak solution to Problem (1.1) with compactly supported initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, $u(\cdot, t)$ is also compactly supported for any $t > 0$, i.e. the solution has finite speed of propagation. This causes the appearance of free boundaries.

b) Let $N = 1$, $m \in (1, 2)$, $s \in (0, 1)$. Then for any $t > 0$ and any $R > 0$, the set $\mathcal{M}_{R,t} = \{x : |x| \geq R, u(x, t) > 0\}$ has positive measure even if u_0 is compactly supported. This is a weak form of infinite speed of propagation. If moreover u_0 is radially symmetric and monotone non-increasing in $|x|$, then we get a clearer result: $u(x, t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.

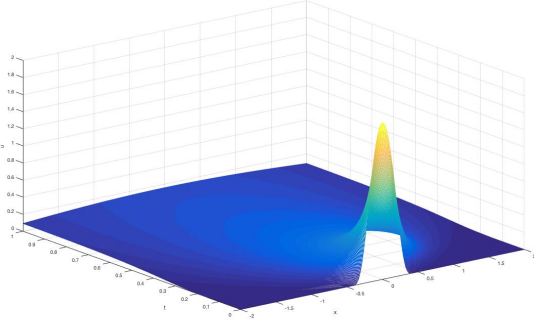
• **The effect of the nonlocal operator on the diffusion.** The parameter $s \in (0, 1)$ plays a crucial role in the the diffusion effects.

a) In the limit $s \rightarrow 1$, we get $u_t = \nabla(u^{m-1} \nabla(-\Delta)^{-1} u)$, which is no more a diffusion equation. This is an interesting problem to be further investigated. When $m = 2$, it has been proved in [28] that the model gives in the limit $s \rightarrow 1$ a "mean field" equation arising in superconductivity and superfluidity. For this equation, the authors obtain uniqueness in the class of bounded solutions, universal bounds and regularity results. To note that Hölder regularity is no more true for the standard class of bounded integrable solutions.

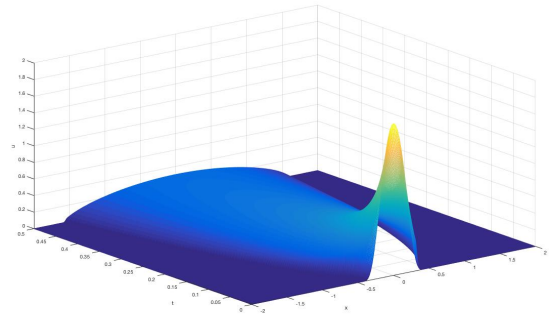
b) When $s \rightarrow 0$ we get $u_t = \nabla(u^{m-1} \nabla u)$ which is the classical Porous Medium Equation $u_t = \frac{1}{m} \Delta u^m$ with $m > 1$. It is known that solutions propagate with finite speed and have C^α regularity.

Such limit processes have not been justified with analytical rigor. We provide some numerical simulations which confirm the behavior of solutions for different values of m and s (see [12, 13]). Figures 2a, 2c, 2e indicate the effect of diffusion in the infinite speed of propagation case. Figures 2b, 2d, 2f indicate the effect of diffusion in the finite speed of propagation case. Note that the larger the s , the

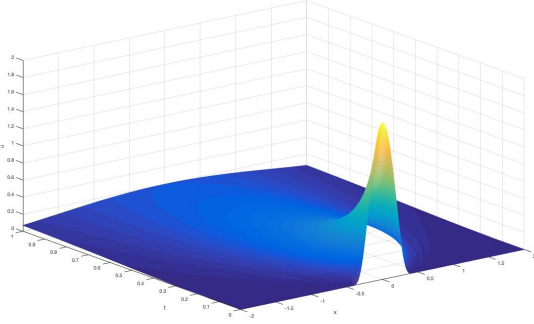
slower is the diffusion velocity.



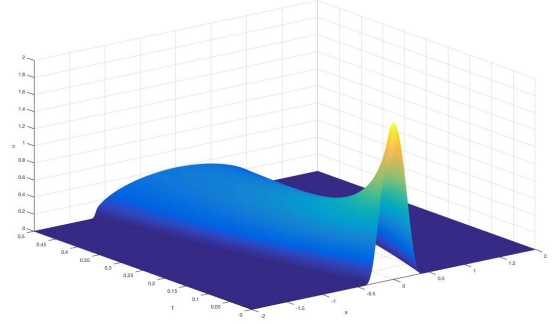
(a) $m = 1.5, s = 0.25$



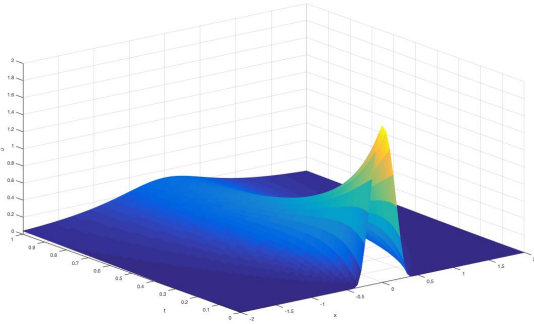
(b) $m = 2, s = 0.25$



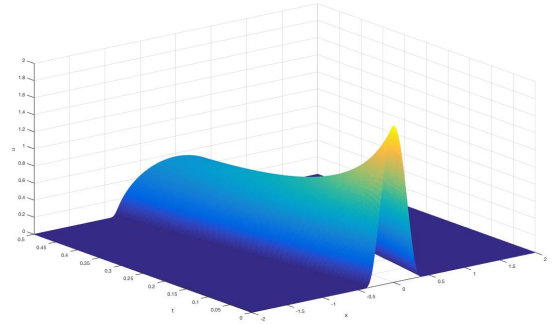
(c) $m = 1.5, s = 0.5$



(d) $m = 2, s = 0.5$



(e) $m = 1.5, s = 0.75$



(f) $m = 2, s = 0.75$

Figure 2: Infinite vs. finite speed of propagation for different pressures

• **Open problems.**

- The problem in a bounded domain with Dirichlet or Neumann data has not been studied.
- Good numerical studies are needed.
- Uniqueness of weak solutions in dimensions $N > 1$.

7 Appendix

7.1 Functional inequalities related to the fractional Laplacian

We recall some functional inequalities related to the fractional Laplacian operator that we used throughout the paper. We refer to [24] for the proofs.

Lemma 7.1 (Stroock-Varopoulos Inequality). *Let $0 < s < 1$, $q > 1$. Then*

$$\int_{\mathbb{R}^N} |v|^{q-2} v (-\Delta)^s v dx \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} |v|^{q/2} \right|^2 dx$$

for all $v \in L^q(\mathbb{R}^N)$ such that $(-\Delta)^s v \in L^q(\mathbb{R}^N)$.

Lemma 7.2 (Generalized Stroock-Varopoulos Inequality). *Let $0 < s < 1$. Then*

$$(7.1) \quad \int_{\mathbb{R}^N} \psi(v) (-\Delta)^s v dx \geq \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \Psi(v) \right|^2 dx$$

whenever $\psi' = (\Psi')^2$.

Theorem 7.3 (Sobolev Inequality). *Let $0 < s < 1$ ($s < \frac{1}{2}$ if $N = 1$). Then*

$$\|f\|_{\frac{2N}{N-2s}} \leq \mathcal{S}_s \left\| (-\Delta)^{s/2} f \right\|_2,$$

where the best constant is given in [5] page 31.

Theorem 7.4 (Nash-Gagliardo-Nirenberg type inequality). *Let $0 < s < 1$ ($s < \frac{1}{2}$ if $N = 1$), $p \geq 1$, $r > 1$, $0 < s < \min\{N/2, 1\}$. Then there exists a constant $C = C(p, r, s, N) > 0$ such that for any $f \in L^p(\mathbb{R}^N)$ with $(-\Delta)^s f \in L^r(\mathbb{R}^N)$ we have*

$$(7.2) \quad \|f\|_{r_2}^{\alpha+1} \leq C \|(-\Delta)^s f\|_r \|f\|_p^\alpha,$$

where $r_2 = \frac{N(rp+r-p)}{r(N-2s)}$, $\alpha = \frac{p(r-1)}{r}$.

7.2 Compactness criteria

Necessary and sufficient conditions of convergence in the spaces $L^p(0, T : B)$ are given by Simon in [29]. We recall now their applications to evolution problems. We consider the spaces $X \subset B \subset Y$ with compact embedding $X \subset B$.

Lemma 7.5. *Let \mathcal{F} be a bounded family of functions in $L^p(0, T : X)$, where $1 \leq p < \infty$ and $\partial \mathcal{F} / \partial t = \{\partial f / \partial t : f \in \mathcal{F}\}$ be bounded in $L^1(0, T : Y)$. Then the family \mathcal{F} is relatively compact in $L^p(0, T : B)$.*

We refer to Rakotoson and Temam [26] for the proof of the following Lemma 7.6 and 7.7.

Lemma 7.6. *Let $(V, \|\cdot\|_V)$, $(H, \|\cdot\|_H)$ two separable Hilbert spaces. Assume that $V \subset H$ with a compact and dense embedding. Consider a sequence $(u_\delta)_{\delta>0}$ converging weakly to a function u in $L^2(0, T : V)$, $T < +\infty$. Then $u_\delta \rightarrow u$ strongly in $L^2(0, T : H)$ if and only if*

(i) $u_\delta \rightharpoonup u(t)$ in H for a.e. t .

(ii) $\lim_{meas(E) \rightarrow 0, E \subset [0, T]} \sup_{\delta > 0} \int_E \|u_\delta(t)\|_H^2 dt = 0$.

Lemma 7.7. *Let H be a separable Hilbert space. Consider u_δ a sequence of functions satisfying the following:*

1) *For almost every $t \in (0, T)$, $\sup_{\delta > 0} \|u_\delta(t)\|_H$ is finite.*

2) *$u \rightharpoonup u$ in $L^2(0, T : H)$.*

3) *There exists a countable set D dense in H such that for all $\psi \in D$, the sequence $g_\psi^\delta(t) = \langle u_\delta(t), \psi \rangle_H$ is relatively compact in $L^1(0, T)$.*

Then, there exists a subsequence $(\delta) = (\delta_D)$ such that $u^\delta(t) \rightharpoonup u(t)$ in H -weak for almost every t .

Combining both lemmas above the following optimal compactness theorem holds.

Theorem 7.8. *Let $(V, \|\cdot\|_V)$, $(H, \|\cdot\|_H)$ two separable Hilbert spaces. Assume that $V \subset H$ with a compact and dense embedding. Consider a sequence $(u_\delta)_{\delta > 0}$ such that*

a) *$u_\delta \rightharpoonup u$ in $L^2(0, T : V)$, $T < +\infty$.*

b) *For almost every $t \in (0, T)$, $\sup_{\delta > 0} \|u_\delta(t)\|_H$ is finite.*

c) *There exists a countable set D dense in H such that for all $\psi \in D$, the sequence $g_\psi^\delta(t) = \langle u_\delta(t), \psi \rangle_H$ is relatively compact in $L^1((0, T))$.*

Then, up to a subsequence, $u_\delta \rightarrow u$ strongly in $L^2(0, T : H)$.

Proof. Weak convergence in $L^2(0, T : V)$ implies weak convergence in $L^2(0, T : H)$, therefore a) implies assumption 2) in Lemma 7.7. By Lemma 7.7 we obtain that, up to a subsequence, $u^\delta(t) \rightharpoonup u(t)$ in H -weak for almost every t . Moreover, the upper bound given by 1) implies (ii) from Lemma 7.6. Then using Lemma 7.6 we obtain that $u_\delta \rightarrow u$ strongly in $L^2(0, T : H)$. \square

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